Geometrically rational real conic bundles and very transitive actions

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Abstract

This article deals with the study the transitivity of the group of automorphisms of real algebraic surfaces. For every real algebraic surface, it is decided when is its group of automorphisms very transitive. Applications are given to the classification of real algebraic models of compact surfaces, with an insight into the geometry of real parts which is a geometry between biregular and birational geometry's, showing several surprising facts about it.

1. Introduction

The group of automorphisms of every complex algebraic variety is small, and even finite in general. Moreover, the group of automorphisms is 3-transitive only if the variety is $\mathbb{P}^1_{\mathbb{C}}$. By contrast, it was recently proved that for a surface $X(\mathbb{R})$ birational to $\mathbb{P}^2_{\mathbb{R}}$, its group of automorphisms acts n-transitively on $X(\mathbb{R})$ for any n. The main goal of this paper is to determine all real algebraic surfaces $X(\mathbb{R})$ having a group of automorphisms which acts very transitively on $X(\mathbb{R})$. See precise definitions and statements below.

Hence, the aim of this paper is to study the action of birational maps on the set of real points of a real algebraic variety. It is worthwhile to point out a common terminological source of confusion about the meaning of what is a *real algebraic variety*, see also the enlightening introduction of [Kol01]. From the point of view of general algebraic geometry, a real variety X is a variety defined over the real numbers, and a morphism is understanding to be defined over all the geometric points. But in most of the texts in real algebraic geometry, the algebraic structure considered corresponds to the one of a neighbourhood of the real points $X(\mathbb{R})$ in the whole complex variety, which is rather the structure of a germ of an algebraic variety defined over $\mathbb{R}$.

From this point of view it is natural to view $X(\mathbb{R})$ as a compact submanifold of $\mathbb{R}^n$ defined by real polynomial equations, where n is some natural integer. Likely, it is natural to say that a map $\psi\colon X(\mathbb{R}) \to Y(\mathbb{R})$ is an *isomorphism* if ψ is induced by a birational map $\Psi\colon X \dashrightarrow Y$ such that Ψ (respectively Ψ^{-1}) is regular at any point of $X(\mathbb{R})$ (respectively of $Y(\mathbb{R})$). In particular, $\psi\colon X(\mathbb{R}) \to Y(\mathbb{R})$ is a diffeomorphism. This notion corresponds to the notion of biregular maps defined in [BCR98, 3.2.6] for the structure of real algebraic variety commonly used in the realm of real algebraic geometry. To distinguish between the Zariski topology and the topology induced by the embedding of $X(\mathbb{R})$ as a topological submanifold of $\mathbb{R}^n$, we will call the later the *Euclidean topology*. In the sequel, topological notions like connectedness or compactness will always refer to the Euclidean topology.

Recall that a real projective surface is rational if it is birationally equivalent to the real projective plane, and that it is geometrically rational if its complexification is birationally equivalent

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to the complex projective plane. The number of connected components is a birational invariant. In particular, if X is a rational projective surface, $X(\mathbb{R})$ is connected.

The paper [HM09a] proves that the group of automorphisms $\mathrm{Aut}\big(X(\mathbb{R})\big)$ acts n-transitively on $X(\mathbb{R})$ for any n and for any rational real algebraic surface X. To study the case when $X(\mathbb{R})$ is not connected, it is necessary to refine the notion of n-transitivity. Indeed, if $X(\mathbb{R})$ has non-homeomorphic connected components, even the group of self-homeomorphisms does not acts 2-transitively.

Definition 0. Let G be a topological group acting continuously on a topological space M. We say that two n-tuples of distinct points $(p_1,\dots,p_n)$ and $(q_1,\dots,q_n)$ are *compatible* if there exists an homeomorphism $\psi\colon M\to M$ such that $\psi(p_i)=q_i$ for each i. Then the action of G on M is said *very transitive* if for any pair of compatible n-tuples of points $(p_1,\dots,p_n)$ and $(q_1,\dots,q_n)$ of M, there exists an element $g\in G$ such that $g(p_i)=q_i$ for each i.

For every real algebraic surface X, a remaining question is to decide when is its group of automorphisms $\mathrm{Aut}\big(X(\mathbb{R})\big)$ big. This question gets finished completely, and this is one of the main result of this paper. We denote by $\#M$ the number of connected components of a compact manifold M.

Theorem 1. *Let X be a nonsingular real projective surface. Then $\mathrm{Aut}\big(X(\mathbb{R})\big)$ has a very transitive action on $X(\mathbb{R})$ if and only if the following holds:*

i) *X is geometrically rational, and*

ii) (a) *$\#X(\mathbb{R})\leqslant 2$, or*
 (b) *$\#X(\mathbb{R})=3$, and there is no pair of homeomorphic connected components, or*
 (c) *$\#X(\mathbb{R})=M_1\sqcup M_2\sqcup M_3$, $M_1\sim M_2\not\sim M_3$, and there is a morphism $\pi\colon X\to\mathbb{P}^1_{\mathbb{R}}$ whose general fibres are rational curves, and an automorphism of $\mathbb{P}^1_{\mathbb{R}}$ which fixes $\pi(M_3)$ and exchanges $\pi(M_1),\pi(M_2)$, or*
 (d) *$\#X(\mathbb{R})=M_1\sqcup M_2\sqcup M_3$, $M_1\sim M_2\sim M_3$, and there is a morphism $\pi\colon X\to\mathbb{P}^1_{\mathbb{R}}$ whose general fibres are rational curves, such that any permutation of the set of intervals $\big\{\pi(M_1),\pi(M_2),\pi(M_3)\big\}$ is realised by an automorphism of $\mathbb{P}^1_{\mathbb{R}}$.*

Furthermore, when $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is not very transitive, it is not even 2-transitive.

This theorem will be proved in Section 9. Notice that in the case when $\#X(\mathbb{R})>3$, either any element of $\mathrm{Aut}\big(X(\mathbb{R})\big)$ preserves a conic bundle structure (Theorem 29), or $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is countable (Corollary 14); thus $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is not 1-transitive.

In the case with three components, Theorem 1 shows that the very transitivity of $\mathrm{Aut}(X(\mathbb{R}))$ is dictated by the exchanges of connected components. In fact, this action is very transitive in a weaker sense.

Let $(p_1,\dots,p_n)$ and $(q_1,\dots,q_n)$ be two compatible n-tuples of points of M such that, for each i, p_i and q_i belong to the same connected component of M. We say that G is very transitive *on each connected component* if for any such a pair, there exists an element $g\in G$ such that $g(p_i)=q_i$ for each i.

Theorem 2. *Let X be a nonsingular geometrically rational real projective surface, and assume that $\#X(\mathbb{R})=3$. Then $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is very transitive on each connected component.*

These two theorems apply to the classification of algebraic models of real surfaces. Until now, we considered $X(\mathbb{R})$ as a submanifold of some $\mathbb{R}^n$. Conversely, let M be a compact $\mathcal{C}^\infty$-manifold. According to the Nash-Tognoli's theorem [Tog73], every such M is diffeomorphic to a nonsingular real algebraic subset of $\mathbb{R}^m$ for some m. Taking the Zariski closure in $\mathbb{P}^m$ and applying Hironaka's resolution of singularities [Hir64], we obtain that M is in fact diffeomorphic to the set of real points $X(\mathbb{R})$ of a nonsingular projective algebraic variety X defined over $\mathbb{R}$. Such a variety X is called an *algebraic model* of M. A natural question is, given M, to classify the algebraic models of M up to isomorphism.

There are several recent results about the question of algebraic models and their groups of automorphisms [BH07, HM09a, HM09b, KM09]. For example, when M is 2-dimensional, and admits a real rational algebraic model, then this rational algebraic model is unique [BH07]. Otherwise speaking, if X and Y are two rational real algebraic surfaces, then $X(\mathbb{R})$ and $Y(\mathbb{R})$ are isomorphic if and only if there are homeomorphic. We extend the classification of real algebraic models to geometrically rational surfaces.

Theorem 3. *Let X, Y be two nonsingular geometrically rational real projective surfaces, and assume that $\#X(\mathbb{R}) \leqslant 2$. Then $X(\mathbb{R})$ is isomorphic to $Y(\mathbb{R})$ if and only if X is birational to Y and $X(\mathbb{R})$ is homeomorphic to $Y(\mathbb{R})$. This is false in general when $\#X(\mathbb{R}) \geqslant 3$.*

Recall that a nonsingular projective surface is minimal if any birational morphism to a nonsingular surface is an isomorphism. We have the following rigidity result about minimal geometrically rational real surfaces.

Theorem 4. *Let X and Y be two minimal geometrically rational real projective surfaces, and assume that either X or Y is non-rational. Then, the following are equivalent:*

- i) *X and Y are birational.*
- ii) *$X(\mathbb{R})$ and $Y(\mathbb{R})$ are isomorphic.*

In the course of this work, we have classified the birational classes of real conic bundles and corrected an error contained in the literature (Theorem 29). As a consequence, we get that the only geometrically rational surfaces $X(\mathbb{R})$ for which equivalence by homeomorphism implies equivalence by isomorphism are the connected ones. In particular, this gives a converse statement to [BH07, Corollary 8.1].

Corollary 5. *Let M be a compact $\mathcal{C}^\infty$-surface. Then M admits a unique geometrically rational model if and only if the following two conditions hold:*

- i) *M is connected, and*
- ii) *M is non-orientable or M is orientable with genus $g(M) \leqslant 1$.*

For M orientable with $g(M) > 1$, there is no result close to some unicity. Thus we can ask what should be the simplest algebraic model for such an M. This question is studied in the forthcoming paper [HM09c].

Another way to measure the size of $\mathrm{Aut}(X(\mathbb{R}))$ was used in [KM09], where it is proved that for any rational surface X, $\mathrm{Aut}\big(X(\mathbb{R})\big) \subset \mathrm{Diff}\big(X(\mathbb{R})\big)$ is dense for the strong topology. For non geometrically rational surfaces, the group $\mathrm{Aut}\big(X(\mathbb{R})\big)$ cannot be dense; as for most of the non-rational geometrically rational surfaces. The cited paper left open the question of density only for some geometrically rational surfaces with 2, 3, 4 or 5 connected components. One by-product of our results is the non-density for most of the surfaces with at least 3 connected components, see Proposition 43.

Let us cite some other papers concerning automorphisms of real projective surfaces. In [RV05], it is proved that $\mathrm{Aut}\big(\mathbb{P}^2(\mathbb{R})\big)$ is generated by linear automorphisms and certain real algebraic automorphisms of degree 5. The paper [HM09b] is devoted to the study of very transitive actions and unicity of models for some kind of singular rational surfaces.

Outline of the article

After giving some notation (in Section 2) that will be used in the article, we recall in Section 3 the classification of minimal geometrically rational real surfaces.

Section 4, which constitutes the technical heart of the paper, is devoted to conic bundles, and especially to the minimal ones. We provide representative elements of isomorphism classes, and explain the links between the conic bundles.

In Section 5, we investigate real surfaces which admit two conic bundles. We show in particular that these are del Pezzo surfaces, and give some description of the possible conic bundles on these surfaces. Section 6 is devoted to the proof of Theorem 4. We firstly correct an existing inaccuracy in the literature, by proving that if two surfaces admitting a conic bundle structure are birational, the birational map may be chosen so that it preserves the conic bundle structures. Then, we strengthen this result to isomorphisms between real parts when the surfaces are minimal, before proving Theorem 4.

In Section 7, we prove that if the real part of a minimal geometrically rational has 2 or 3 connected components, its group of automorphisms is very transitive on each connected component. In Section 8, we do the same work with non-minimal surfaces. We show how to separate infinitely near points, which is certainly one of the most counter-intuitive behaviour of our geometry, and was first observed in [BH07] for rational surfaces. We also obtain the unicity of models in many cases.

Then, in Section 9, we use all the results of the previous sections, to give the proof of the main results stated in the introduction (except Theorem 4, already proved in Section 6).

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2. Notation

In the sequel, by a variety we will mean an algebraic variety, which may be real or complex (i.e. defined over $\mathbb{R}$ or $\mathbb{C}$). If the converse is not expressly stated all our varieties will be projective and all our surfaces will be nonsingular and geometrically rational (i.e. rational over $\mathbb{C}$).

Recall that a real variety X may be identified with a pair (S,σ), where S is a complex variety and σ is an anti-holomorphic involution on S; by abuse of notation we will write $X=(S,\sigma)$. Then, $S(\mathbb{C})=X(\mathbb{C})$ denotes the set of complex points of the variety, and $X(\mathbb{R})=S(\mathbb{C})^\sigma$ is the set of real points. A point $p\in X$ may be real (if it belongs to $X(\mathbb{R})$), or imaginary (if it belongs to $X(\mathbb{C})\backslash X(\mathbb{R})$). If $X(\mathbb{R})$ is non empty (which will be the case for all our surfaces), then $\mathrm{Pic}(X)\cong\mathrm{Pic}(S)^\sigma$, [Sil89, I.(4.5)]. As we only work with regular surfaces (i.e. $q(X)=q(S)=0$), the Picard group is isomorphic to the Néron-Severi group, and $\rho(S)$ and $\rho(X)$ will denote the rank of $\mathrm{Pic}(S)$ and $\mathrm{Pic}(X)$ respectively. Recall that $\rho(X)\leqslant\rho(S)$. We denote by $K_X\in\mathrm{Pic}(X)$ the canonical class, which may be identified with K_S. The intersection of two divisors of $\mathrm{Pic}(S)$ or $\mathrm{Pic}(X)$ will always denote the usual intersection in $\mathrm{Pic}(S)$.

We will use the classical notions of morphisms, rational maps, isomorphisms and automorphisms between real or complex varieties. Moreover, if X_1 and X_2 are two real varieties, an isomorphism *between real parts* $X_1(\mathbb{R})\xrightarrow{\psi}X_2(\mathbb{R})$ is a birational map $\psi\colon X_1\dashrightarrow X_2$ such that ψ (respectively ψ^{-1}) is regular at any point of $X_1(\mathbb{R})$ (respectively of $X_2(\mathbb{R})$). This endows $X_1(\mathbb{R})$ with a structure of a germ of algebraic variety defined over $\mathbb{R}$ (as in [BCR98, 3.2.6]), whereas the structure of X_1 is the one of an algebraic variety.

Considering geometry's on algebraic variety defined over $\mathbb{R}$, this notion of isomorphism between real parts gives an intermediate geometry in between the biregular geometry and the birational geometry. For example, let $\alpha\colon X_1(\mathbb{R})\to X_2(\mathbb{R})$ be an isomorphism, and $\varepsilon\colon Y_1\dashrightarrow X_1$, $\eta\colon Y_2\dashrightarrow X_2$

be two birational maps; the map $\psi := \varepsilon^{-1}\alpha\eta$ is a well-defined birational map. Then ψ can be an isomorphism $Y_1(\mathbb{R}) \to Y_2(\mathbb{R})$ even if nor ε, nor η is an isomorphism between real parts. In the same vein, let $\alpha\colon X_1(\mathbb{R}) \to X_2(\mathbb{R})$ be an isomorphism, and let $\eta_1\colon Y_1 \to X_1$ and $\eta_2\colon Y_2 \to X_2$ be two birational morphisms which are the blow-ups of only real points (which may be proper or infinitely near points of X_1 and X_2). If α sends the points blown-up by η_1 on the points blown-up by η_2, then $\beta = (\eta_2)^{-1}\alpha\eta_1\colon Y_1(\mathbb{R}) \to Y_2(\mathbb{R})$ is an isomorphism.

Using Aut and Bir to denote respectively the group of automorphisms and birational self-maps of a variety, we have the following inclusions for the groups associated to $X = (S, \sigma)$:

$$\begin{array}{ccccc} \mathrm{Aut}(S) & & \subset & & \mathrm{Bir}(S) \\ \cup & & & & \cup \\ \mathrm{Aut}(X) & \subset & \mathrm{Aut}\big(X(\mathbb{R})\big) & \subset & \mathrm{Bir}(X)\ . \end{array}$$

By $\mathbb{P}^n$ we mean the projective n-space, which may be complex or real depending on the context. It is unique as a complex variety – written $\mathbb{P}^n_{\mathbb{C}}$. However, as a real variety, $\mathbb{P}^n$ may either be $\mathbb{P}^n_{\mathbb{C}}$ endowed with the standard anti-holomorphic involution, written $\mathbb{P}^n_{\mathbb{R}}$, or only when n is odd, $\mathbb{P}^n_{\mathbb{C}}$ with a special involution with no real points, written $(\mathbb{P}^n, \emptyset)$. To lighten notation, and since we never speak about $(\mathbb{P}^1, \emptyset)(\mathbb{R})$ we write $\mathbb{P}^1(\mathbb{R})$ for $\mathbb{P}^1_{\mathbb{R}}(\mathbb{R})$.

3. Minimal surfaces and minimal conic bundles

The aim of this section is to reduce our study of geometrically rational surfaces to surfaces which admit a minimal conic bundle structure.

Definition 6. A surface X is said to be *minimal* if any birational morphism from X to a (nonsingular) surface is an isomorphism.

When X is real, this definition is equivalent to say that there is no real (-1)-curve and no pair of disjoint conjugate imaginary (-1)-curves on X.

Let us precise the notion of conic bundle. Since we only deal with geometrically rational surfaces, the basis of our conic bundles is always geometrically rational.

Definition 7. A *conic bundle* is a pair (X, π) where X is a surface and π is a morphism $X \to \mathbb{P}^1$, where any fibre of π is isomorphic to a plane conic.

Note that if (X, π) is complex, a general fibre of π is isomorphic to $\mathbb{P}^1_{\mathbb{C}}$, and a singular fibre consists of the union of two intersecting lines which are (-1)-curves of X (the double line is not allowed, the surface X being nonsingular). If (X, π) is real, a fibre over a real point of $\mathbb{P}^1$ may be isomorphic to $\mathbb{P}^1_{\mathbb{R}}$, to $(\mathbb{P}^1, \emptyset)$, or to a singular fibre with two irreducible components which can be two real components, or two conjugated imaginary components intersecting in 1 real point.

We will assume in the sequel that if X is real, then the basis is $\mathbb{P}^1_{\mathbb{R}}$ (and not $(\mathbb{P}^1, \emptyset)$). This avoids certain conic bundles with no real points.

Definition 8. If (X, π) and (X', π') are two conic bundles, a *birational map of conic bundles* $\psi\colon (X, \pi) \dashrightarrow (X', \pi')$ is a birational map $\psi\colon X \dashrightarrow X'$ such that there exists an automorphism α of $\mathbb{P}^1$ with $\pi' \circ \phi = \pi \circ \alpha$.

This notion specialises to birational morphisms and to automorphisms of conic bundles. We denote by $\mathrm{Aut}(X, \pi)$ (respectively $\mathrm{Bir}(X, \pi)$) the group of automorphisms (respectively birational self-maps) of the conic bundle (X, π). Observe that $\mathrm{Aut}(X, \pi) = \mathrm{Aut}(X) \cap \mathrm{Bir}(X, \pi)$. Similarly, when (X, π) is real we define by $\mathrm{Aut}(X(\mathbb{R}), \pi)$ the group $\mathrm{Aut}\big(X(\mathbb{R})\big) \cap \mathrm{Bir}(X, \pi)$.

DEFINITION 9. A conic bundle (X,π) is said to be *minimal* if any birational morphism of conic bundles $(X,\pi)\to(X',\pi')$ is an isomorphism.

Note that a real conic bundle (X,π) is minimal if and only if the two irreducible components of any real singular fibre of π are imaginary. Compare to the complex case where (X,π) is minimal if and only if there is no singular fibre.

Any geometrically rational real surface Y is obtained by a finite sequence of blow-ups (centred at a real point or at a pair of conjugate imaginary points) from a minimal real surface X. The following classical theorem describes the possibilities for the surface X.

Recall that a surface X is a del Pezzo surface if the anti-canonical divisor $-K_X$ is ample. The same definition applies for X real or complex.

THEOREM 10. *If X is a minimal geometrically rational real surface such that $X(\mathbb{R})\neq\emptyset$, then one and exactly one of the following holds:*

i) *X is rational: it is isomorphic to $\mathbb{P}^2_{\mathbb{R}}$, to the quadric $Q_0:=\{(x:y:z:t)\in\mathbb{P}^3_{\mathbb{R}}\mid x^2+y^2+z^2=t^2\}$, or to a real Hirzebruch surface $\mathbb{F}_n$, $n\neq1$;*
ii) *X is a del Pezzo surface of degree 1 or 2 with $\rho(X)=1$;*
iii) *there exists a minimal conic bundle structure $\pi\colon X\to\mathbb{P}^1$ with an even number of singular fibres $2r\geqslant4$. Moreover, $\rho(X)=2$.*

Remark 11. If (S,σ) is a minimal geometrically rational real surface such that $S(\mathbb{C})^\sigma=\emptyset$, then S is an Hirzebruch surface of even index.

Proof. Follows from the work of Comessatti [Com12], (see also [Mani67], [Isk79], [Sil89, Chap. V], or [Kol97]). □

PROPOSITION 12 Topology of the real part. *In each case of the former theorem, we have:*

i) *X is rational if and only if $X(\mathbb{R})$ is connected. When X is moreover minimal, then $X(\mathbb{R})$ is homeomorphic to one of the following: the real projective plane, the sphere, the torus, or the Klein bottle.*
ii) *When X is a minimal del Pezzo surface of degree 1, it satisfies $\rho(X)=1$, and $X(\mathbb{R})$ is the disjoint union of one real projective plane and 4 spheres. If X is a minimal del Pezzo surface of degree 2 with $\rho(X)=1$, then $X(\mathbb{R})$ is the disjoint union of 4 spheres.*
iii) *If X is non-rational and is endowed with a minimal conic bundle with $2r$ singular fibres, then $X(\mathbb{R})$ is the disjoint union of r spheres, $r\geqslant2$.*

Proof. For the first assertion, see [Sil89, Corollary VI(6.5)], for the other ones, see e.g. [Sil89, Chap. V] or [Kol97]. □

The proofs of Theorems 1 and 4 will split into the cases listed in Theorem 10. The rational case is treated in [HM09a]. The next proposition states the case when X is a minimal del Pezzo surface with $\rho=1$. The remaining part of the paper is mainly devoted to the case when X is endowed with a minimal real conic bundle.

PROPOSITION 13. *Let X,Y be two minimal geometrically rational real surfaces. Assume that X is not rational and satisfies $\rho(X)=1$ (but $\rho(Y)$ may be equal to 1 or 2).*

i) *If X is a del Pezzo surface of degree 1, then any birational map $X\dashrightarrow Y$ is an isomorphism. In particular,*

$$\operatorname{Aut}(X)=\operatorname{Aut}\big(X(\mathbb{R})\big)=\operatorname{Bir}(X)\ .$$

ii) *If X is a del Pezzo surface of degree 2, X is birational to Y if and only X is isomorphic to Y. Moreover, all the base-points of the elements of* $\mathrm{Bir}(X)$ *are real, and*

$$\mathrm{Aut}(X) = \mathrm{Aut}\big(X(\mathbb{R})\big) \subsetneq \mathrm{Bir}(X) \ .$$

Proof. Assume the existence of a birational map $\psi\colon X \dashrightarrow Y$. If ψ is not an isomorphism, we decompose ψ into elementary links

$$X = X_0 \overset{\psi_1}{\dashrightarrow} X_1 \overset{\psi_2}{\dashrightarrow} \cdots \overset{\psi_{n-1}}{\dashrightarrow} X_{n-1} \overset{\psi_n}{\dashrightarrow} X_n = Y$$

as in [Isk96, Theorem 2.5]. It follows from the description of the links of [Isk96, Theorem 2.6] that for any link $\psi_i\colon X_{i-1} \dashrightarrow X_i$, X_{i-1} and X_i are isomorphic del Pezzo surfaces of degree 2, and that ψ_i is equal to $\beta\eta\alpha\eta^{-1}$, where η is the blow-up $X' \to X_{i-1}$ of a real point of X_{i-1}, X' is a del Pezzo surface of degree 1, $\alpha \in \mathrm{Aut}(X')$ is the Bertini involution of the surface, and $\beta\colon X_{i+1} \to X_i$ is an isomorphism.

Therefore, Y is isomorphic to X. Moreover, if X has degree 1, ψ is an isomorphism. If X has degree 2, ψ is decomposed into conjugates of Bertini involutions, so each of its base-points is real. This proves that if $\psi \in \mathrm{Aut}\big(X(\mathbb{R})\big)$ then $\psi \in \mathrm{Aut}(X)$. Furthermore, conjugates of Bertini involutions belong to $\mathrm{Bir}(X)$ but not to $\mathrm{Aut}(X) = \mathrm{Aut}\big(X(\mathbb{R})\big)$. □

Corollary 14. *Let X_0 be a minimal non-rational geometrically rational real surface with $\rho(X_0) = 1$, and let $\eta\colon X \to X_0$ be a birational morphism.*

Then, $\mathrm{Aut}\big(X(\mathbb{R})\big)$ *is countable. Moreover, if X_0 is a del Pezzo surface of degree 1, then* $\mathrm{Aut}\big(X(\mathbb{R})\big)$ *is finite.*

Proof. Without changing the isomorphism class of $X(\mathbb{R})$ we may assume that η is the blow-up of only real points (which may belong to X_0 as proper or infinitely near points). Since any base-point of any element of $\mathrm{Bir}(X_0)$ is real (Proposition 13), the same is true for any element of $\mathrm{Bir}(X)$. In particular, $\mathrm{Aut}\big(X(\mathbb{R})\big) = \mathrm{Aut}(X)$. The group $\mathrm{Aut}(X)$ acts on $\mathrm{Pic}(X) \cong \mathbb{Z}^n$, where $n = \rho(X) \geqslant 1$. This action gives rise to an homomorphism $\theta\colon \mathrm{Aut}(X) \to \mathrm{GL}(n, \mathbb{Z})$. Let us prove that θ is injective. Indeed, if $\alpha \in \mathrm{Ker}(\theta)$, then α is conjugate by η to an element of $\alpha_0 \in \mathrm{Aut}(X_0)$ which acts trivially on $\mathrm{Pic}(X_0)$. Writing S_0 the complex surface obtaining by forgetting the real structure of X_0, S_0 is the blow-up of 7 or 8 points in general position of $\mathbb{P}^2_{\mathbb{C}}$. Thus $\alpha_0 \in \mathrm{Aut}(X_0) \subset \mathrm{Aut}(S_0)$ is the lift of an automorphism of $\mathbb{P}^2_{\mathbb{C}}$ which fixes 7 or 8 points, no 3 collinear, hence is the identity.

The morphism θ is injective, and this shows that $\mathrm{Aut}\big(X(\mathbb{R})\big) = \mathrm{Aut}(X)$ is countable. Moreover, if X_0 is a del Pezzo surface of degree 1, then $\mathrm{Bir}(X_0) = \mathrm{Aut}(X_0)$ (by Proposition 13). Since $\mathrm{Aut}(X_0)$ is finite, $\mathrm{Aut}\big(X(\mathbb{R})\big) \subset \mathrm{Bir}(X)$ is also finite. □

4. Minimal and exceptional conic bundles

Definition 15. If (X, π) is a real conic bundle, $I(X, \pi) \subset \mathbb{P}^1(\mathbb{R})$ denotes the image by π of the set $X(\mathbb{R})$ of real points of X.

It is well-known that $I(X, \pi)$ is the union of a finite number of intervals (which may be $\emptyset$ or $\mathbb{P}^1(\mathbb{R})$), and that it determines the birational class of (X, π). In the next section, we will prove that in fact $I(X, \pi)$ determines the birational class of X, and thus that $I(X)$ is well-defined. In this section, we study the real conic bundles, and especially the minimal ones. We prove that $I(X, \pi)$ also determines the equivalence class of $(X(\mathbb{R}), \pi)$ among the minimal conic bundles, and give the proof of Theorem 4 in the case of conic bundles (Corollary 24). Doing this, we will give proofs of the well-known facts on $I(X, \pi)$ cited above.

Lemma and Definition 16. *Let (X,π) be a real minimal conic bundle. The following conditions are equivalent:*

i) *There exists a section s such that s and $\bar{s}$ do not intersect.*

ii) *There exists a section s such that $s^2=-r$, where $2r$ is the number of singular fibres.*

If any of these conditions occur, we say that (X,π) is exceptional.

Proof. Let s be a section satisfying one of the two conditions. Denote by (S,π) the complex conic bundle obtained by forgetting the real structure of (X,π), and by $\eta: X\to\mathbb{F}_m$ the birational map which contracts in any singular fibre of π the irreducible component which does not intersect s. If s satisfies condition $i)$, $\eta(\bar{s})$ and $\eta(s)$ are two sections of $\mathbb{F}_m$ which do not intersect, so they have self-intersections $-m$ and m. This means that $s^2=\bar{s}^2=-m$ and that the number of singular fibres is $2m$, and implies $ii)$. Conversely, if s satisfies $ii)$, $\eta(s)$ and $\eta(\bar{s})$ are sections of $\mathbb{F}_m$ of self-intersection $-r$ and r. If these two sections are distinct, they do not intersect, which means that s and $\bar{s}$ do not intersect. If $\eta(s)=\eta(\bar{s})$, we have $r=0$, and $X=(\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}},\sigma)$ for a certain anti-holomorphic involution σ. We may thus choose another section s' of self-intersection 0 which is imaginary. □

Remark 17. The definition of exceptional conic bundles was introduced in [DI06] and [Bla09b] for *complex* conic bundles endowed with an *holomorphic* involution. If (S,π) is an exceptional complex conic bundle with at least 4 singular fibres, $\mathrm{Aut}(S,\pi)=\mathrm{Aut}(S)$ is a maximal algebraic subgroup of $\mathrm{Bir}(S)$ [Bla09b].

We now describe the explicit form of exceptional conic bundles, which will be useful in the sequel.

Lemma 18. *Let (X,π) be an exceptional real conic bundle. Then, there exists an affine real variety $A\subset X$ isomorphic to the affine surface of $\mathbb{R}^3$ given by*

$$y^2+z^2=Q(x),$$

where Q is a real polynomial with only simple roots, all real. Moreover, $\pi|_A\colon A\to\mathbb{P}^1_{\mathbb{R}}$ is the projection $(x,y,z)\mapsto(x:1)$, and $I(X,\pi)$ is the closure of $\{(x:1)\in\mathbb{P}^1_{\mathbb{R}}\mid Q(x)\geqslant 0\}$.

Furthermore, if $f=\pi^{-1}((1:0))\subset X$ is a nonsingular fibre, the singular fibres of π are those of the points $\{(x:1)\mid Q(x)=0\}$ and the inclusion $A\to X$ is an isomorphism $A(\mathbb{R})\to(X\backslash f)(\mathbb{R})$. In particular, if $(1:0)\notin I(X,\pi)$, the inclusion yields an isomorphism $A(\mathbb{R})\to X(\mathbb{R})$.

Proof. Denote by $2r$ the number of singular fibres of π (which is even, see Lemma 16).

Assume first that $r=0$, which implies that (X,π) is a real form of $(\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}},\mathrm{pr}_1)$, hence is isomorphic to $(\mathbb{P}^1_{\mathbb{R}}\times\mathbb{P}^1_{\mathbb{R}},\mathrm{pr}_1)$ or to $(\mathbb{P}^1_{\mathbb{R}}\times(\mathbb{P}^1,\emptyset),\mathrm{pr}_1)$, see convention before Definition 8. Taking $Q(x)=1$ or $Q(x)=-1$ gives the result.

Assume now that $r>0$, and denote by s and $\bar{s}$ two conjugate imaginary sections of π of self-intersection $-r$. Changing π by an automorphism of $\mathbb{P}^1$, we can assume that $(1:0)$. The singular fibres of π are above the points $(a_1:1),\dots,(a_{2r}:1)$, where the a_i are distinct real numbers. Let $J=(J_1,J_2)$ be a partition of $\{a_1,\dots,a_{2r}\}$ into two sets of r points. Let η be the birational morphism (not defined over $\mathbb{R}$) which contracts the irreducible component of $\pi^{-1}((a_i:1))$ which intersects s if $a_i\in J_1$ and the component which intersects $\bar{s}$ if $a_i\in J_2$. Then, the images of s and $\bar{s}$ are two sections of self-intersection 0. Thus we may assume that η is a birational morphism of conic bundles $(S,\pi)\to(\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}},\mathrm{pr}_1)$, where S is the complex surface obtained by forgetting the real structure of X, pr_1 is the projection on the first factor, and where $\eta(s)$ and $\eta(\bar{s})$ are equal to $\mathbb{P}^1_{\mathbb{C}}\times(0:1)$ and $\mathbb{P}^1_{\mathbb{C}}\times(1:0)$.

We write $P_1(x_1,x_2) = \prod_{a\in J_1}(x_1 - ax_2)$ and $P_2(x_1,x_2) = \prod_{a\in J_2}(x_1 - ax_2)$, and denote by α and σ the self-maps of S, which are the lifts by η of the following self-maps of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$:

$$\begin{aligned} \alpha' &\colon \big((x_1:x_2),(y_1:y_2)\big) \dashrightarrow \big((x_1:x_2),(-y_2\cdot P_1(x_1,x_2) : y_1 \cdot P_2(x_1,x_2)\big), \\ \sigma' &\colon \big((x_1:x_2),(y_1:y_2)\big) \dashrightarrow \big((\overline{x_1}:\overline{x_2}),(-\overline{y_2}\cdot P_1(\overline{x_1},\overline{x_2}) : \overline{y_1} \cdot P_2(\overline{x_1},\overline{x_2})\big). \end{aligned}$$

The map α' is a birational involution of $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, which is defined over $\mathbb{R}$, and whose base-points are precisely the $2r$ points $\{((x:1),(0:1)) \mid x \in J_1\} \cup \{((x:1),(1:0)) \mid x \in J_2\}$ blown-up by η. Since α' is an involution and η is the blow-up of all of its base-points, $\alpha = \eta^{-1}\alpha'\eta$ is an automorphism of S, which belongs to $\mathrm{Aut}(S,\pi)$. In consequence, σ is an anti-holomorphic involution of S.

Denote by σ_X the anti-holomorphic involution on S which gives the real structure of X. The map $\sigma_X \circ \sigma^{-1}$ belongs to $\mathrm{Aut}(S,\pi)$ and acts trivially on the basis, since σ and σ_X have the same action on the basis. Moreover, since both σ_X and σ exchange the irreducible components of each singular fibre, $\sigma_X \circ \sigma^{-1}$ preserves any curve contracted by η and is therefore the lift by η of $\beta\colon \big((x_1 : x_2),(y_1:y_2)\big) \mapsto \big((x_1:x_2),(\mu y_1 : y_2)\big)$ for some $\mu \in \mathbb{C}^*$. It follows that $\sigma'_X = \eta \circ \sigma_X \circ \eta^{-1} = \beta \circ \sigma'$ is the map

$$\sigma'_X \colon \big((x_1:x_2),(y_1:y_2)\big) \dashrightarrow \big((\overline{x_1}:\overline{x_2}),(-\mu\cdot\overline{y_2}P_1(\overline{x_1},\overline{x_2}) : \overline{y_1}P_2(\overline{x_1},\overline{x_2}))\big).$$

Let us write $Q(x) = -\mu P_1(x,1)P_2(x,1)$, denote by $B \subset \mathbb{C}^3$ the affine hypersurface of equation $y^2+z^2 = Q(x)$, and by $\pi_B\colon B \to \mathbb{P}^1$ the map $(x,y,z) \mapsto (x:1)$. Let $A = (B,\sigma_B)$, where σ_B sends (x,y,z) onto $(\bar{x},\bar{y},\bar{z})$. Denote by $\theta\colon B \dashrightarrow \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ the map that sends (x,y,z) onto $\big((x:1),(y-\mathbf{i}z : P_2(x,1))\big)$ if $P_2(x,1) \neq 0$ and onto $\big((x:1),(-\mu P_1(x,1) : y+\mathbf{i}z)\big)$ if $P_1(x,1)\neq 0$. Then θ is a birational morphism, and θ^{-1} sends $\big((x_1:x_2),(y_1:y_2)\big)$ on

$$\left(\frac{x_1}{x_2}, \frac{1}{2}\left(\frac{y_1}{y_2}P_2(x_1,x_2) - \frac{y_2}{y_1}\mu P_1(x_1,x_2)\right), \frac{\mathbf{i}}{2}\left(\frac{y_1}{y_2}P_2(x_1,x_2) + \frac{y_2}{y_1}\mu P_1(x_1,x_2)\right)\right).$$

Observe that $\sigma'_X\theta = \sigma_B\theta$. In consequence, $\psi = \eta^{-1}\circ\theta$ is a real birational map $A \dashrightarrow X$.

Moreover, ψ is an isomorphism from B to the complement in S of the union of $\pi^{-1}((1:0))$ and the pull-back by η of $\mathbb{P}^1 \times (0:1)$ and $\mathbb{P}^1\times(1:0)$. Indeed let $x_0 \in \mathbb{C}$. If $x_0 \in \mathbb{C}$ is such that $Q(x_0)\neq 0$, then θ restricts to an isomorphism from $\pi_B^{-1}((x_0:1))$ to $\{((x_0:1),(y_1:y_2)) \in \mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}} \mid y_1y_2 \neq 0\} \cong \mathbb{C}^*$. If $Q(x_0)=0$, then $x_0 \in J_1\cup J_2$, and the fibre $\pi_B^{-1}((x_0:1))$ consists of two lines of $\mathbb{C}^2$ which intersect, given by $y = \mathbf{i}z$ and $y = -\mathbf{i}z$. If $x_0 \in J_1$, then the line $y+\mathbf{i}z = 0$ is sent isomorphically by θ onto the fibre $\{((x_0:1),(y_1:y_2)) \in \mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}} \mid y_2 \neq 0\} \cong \mathbb{C}^*$, and the line $y - \mathbf{i}z$ is contracted on the point $((x_0:1),(0:1))$. The map ψ sends thus isomorphically $\pi_B^{-1}((x_0:1))$ onto the fibre $\pi^{-1}((x_0:1))$ minus the two points corresponding to the two sections of self-intersection $-r$. The situation when $x_0 \in J_2$ is similar.

The map ψ is therefore an inclusion $A \to X$ and, by construction, it satisfies all the properties stated in the lemma. □

Lemma 19. *Let (Y,π_Y) be a minimal real conic bundle such that π_Y has at least one singular fibre. There exists an* exceptional *real conic bundle (X,π_X) and an isomorphism $\psi\colon Y(\mathbb{R}) \to X(\mathbb{R})$ such that $\pi_X\circ\psi = \pi_Y$.*

Remark 20. The result is false without the assumption on the number of singular fibres. Consider for example $Y = \mathbb{F}_3(\mathbb{R})$, whose real part is homeomorphic to the Klein bottle. Indeed, any exceptional conic bundle with no singular fibres is a real form of $(\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}, \mathrm{pr}_1)$, and thus has a real part either empty or homeomorphic to the torus $S^1\times S^1$.

Proof of Lemma 19. Take a section s of π_Y. If s intersects its conjugate $\bar{s}$ into a real point p (respectively into a pair of imaginary points q_1 and q_2), then blow-up the point p (respectively q_1

and q_2), and contract the strict transform of the fibre of the blown-up point(s). Repeating this process, we obtain a minimal real conic bundle (Z, π_Z) and a birational map $\phi\colon Y \dashrightarrow Z$ such that $\pi_Z \circ \phi = \pi_Y$ and $\phi(s)$ does not intersect its conjugate.

If all the base-points of ϕ are imaginary, we set $\psi = \phi$ and $(X, \pi_X) = (Z, \pi_Z)$. Otherwise, by induction on the number of real base-points of ϕ, it suffices to prove the existence of ψ when ϕ is an elementary link centered at only one real point.

Denote by $q \in Z$ the real point which is the base-point of ϕ^{-1}. Since π_Y has at least one singular fibre, this is also the case for π_Z, and thus $I(Z, \pi_Z)$ is not the whole $\mathbb{P}^1(\mathbb{R})$ (By Lemma 18). We may thus assume that $(1:0) \notin I(Z, \pi_Z)$, that $\pi_Z(q) = (1:1)$, and that the interval of $I(Z, \pi_Z)$ which contains $\pi_Z(q)$ is $\{(x:1) \in \mathbb{P}^1_{\mathbb{R}} \mid 0 \leqslant x \leqslant a\}$ for some $a \in \mathbb{R}$, $a > 1$. Take the affine surface $A \subset Z$ given by Lemma 18, which is isomorphic to $y^2 + z^2 = Q(x)$ for some polynomial Q. Then, $Q(0) = Q(a) = 0$ and $Q(x) > 0$ for $0 < x < a$, and we may assume that $Q(1) = 1$. Denote by s the section of $\pi_Z\colon Z \to \mathbb{P}^1_{\mathbb{R}}$ given locally by $y + \mathbf{i}z = \mathbf{i}x^n$, for some positive integer n. Its conjugate is given by $y - \mathbf{i}z = -\mathbf{i}x^n$, or $y + \mathbf{i}z = Q(x)/(-\mathbf{i}x^n)$. Thus, s intersects $\bar{s}$ at some real point $p \in Z$, its image $x = \pi_Z(p)$ satisfies $Q(x)/(-\mathbf{i}x^n) = \mathbf{i}x^n$, or $Q(x) = x^{2n}$. Taking n large enough, this can only happen when $x = 0$ or $x = 1$. The first possibility cannot occur since a section does not pass through the singular point of a singular fibre. Thus, s intersects $\bar{s}$ at only one real point, which is q. In consequence, the strict pull-back by ϕ of s is a section of Y which intersects its conjugate at only imaginary points. This shows that $(Y(\mathbb{R}), \pi_Y)$ is isomorphic to an exceptional real conic bundle (X, π_X). □

We can now give the well-known description of $I(X, \pi)$ announced before.

Corollary 21. *Let (X, π) be a real conic bundle.*

i) *$I(X, \pi)$ is a finite union of closed intervals of $\mathbb{P}^1_{\mathbb{R}}$ (which may be $\emptyset$ or $\mathbb{P}^1_{\mathbb{R}}$).*
ii) *The r connected components of $X(\mathbb{R})$ surject by π on r closed intervals of $\mathbb{P}^1(\mathbb{R})$.*
iii) *If (X, π) is minimal, the images by π of the $2r$ singular fibres of π are the boundary points of the intervals of $I(X, \pi)$. Moreover, $(K_X)^2 = 8 - 2r$.*

Proof. We may assume that (X, π) is minimal. If there is no singular fibre, (X, π) is a real form of some Hirzebruch surface and $I(X, \pi)$ is $\emptyset$ or $\mathbb{P}^1_{\mathbb{R}}$. Otherwise, we may assume – according to Lemma 19 – that (X, π) is exceptional. The result is then a direct consequence of Lemma 18, except the calculation on $(K_X)^2$. Denoting by S the complex surface obtained by forgetting the real structure on X, we have $(K_S)^2 = (K_X)^2$. Blowing-down $2r$ (-1)-curves, one in each singular fibre, we get a morphism from S to an Hirzebruch surface $\mathbb{F}_m$. Since $(K_{\mathbb{F}_m})^2 = 8$, we have $(K_S)^2 = 8 - 2r$. □

Proposition 22. *Let (X, π_X) and (Y, π_Y) be two minimal real conic bundles, and assume that either π_X or π_Y has at least one singular fibre. Then, the following are equivalent:*

i) *$I(X, \pi_X) = I(Y, \pi_Y)$;*
ii) *there exists a birational map $\varphi\colon X \dashrightarrow Y$ such that $\pi_Y \circ \varphi = \pi_X$;*
iii) *there exists an isomorphism $\varphi\colon X(\mathbb{R}) \to Y(\mathbb{R})$ such that $\pi_Y \circ \varphi = \pi_X$.*

Proof. We prove now *ii)* $\Rightarrow$ *i)*. The birational map of *ii)* decomposes into elementary links of conic bundles; in particular, the singular fibres of π_X and π_Y are above the same points of $\mathbb{P}^1(\mathbb{R})$. By Corollary 21, we get $I(X, \pi_X) = I(Y, \pi_Y)$.

iii) $\Rightarrow$ *ii)* is obvious; it remains to prove *i)* $\Rightarrow$ *iii)*. By Lemma 19, we may assume that both (X, π_X) and (Y, π_Y) are exceptional. We may now assume that the fibre over $(1:0)$ is not singular and use Lemma 18: let $A_X \subset X$ and $B_X \subset Y$ be the affine surfaces given by the lemma, with equations $y^2 + z^2 = Q_X(x)$ and $y^2 + z^2 = Q_Y(x)$ respectively. Since $I(X, \pi_X) = I(Y, \pi_Y)$, $Q_Y(x) =$

$\lambda Q_X(x)$ for some positive $\lambda \in \mathbb{R}$. The map $(x,y,z) \mapsto (x, \sqrt{\lambda}y, \sqrt{\lambda}z)$ then yields an isomorphism $(X(\mathbb{R}), \pi_X) \to (Y(\mathbb{R}), \pi_Y)$. □

The above result implies the next two corollaries. Part i) of Corollary 23 is an important result, due to Comessatti [Com12] (see also [Kol97, Theorem 4.5]). The other parts strengthen this result.

Corollary 23. *Let (X,π) and (X',π') be two real conic bundles.*

i) *(X,π) and (X',π') are birational if and only if there exists an automorphism of $\mathbb{P}^1_\mathbb{R}$ that sends $I(X,\pi)$ on $I(X',\pi')$.*
ii) *Assume that (X,π) and (X,π') are minimal. Then $(X(\mathbb{R}),\pi)$ and $(X'(\mathbb{R}),\pi')$ are isomorphic if and only if there exists an automorphism of $\mathbb{P}^1_\mathbb{R}$ that sends $I(X,\pi)$ on $I(X',\pi')$.*

Proof. We may assume that (X,π) and (X',π') are minimal. Then, the results follows directly from Proposition 22. □

Corollary 24. *Let (X,π_X) and (Y,π_Y) be two minimal conic bundles. Then, the following are equivalent:*

i) *$(X(\mathbb{R}),\pi_X)$ and $(Y(\mathbb{R}),\pi_Y)$ are isomorphic;*
ii) *(X,π_X) is birational to (Y,π_Y) and $X(\mathbb{R})$ is isomorphic to $Y(\mathbb{R})$.*

Proof. The implication *i)* $\Rightarrow$ *ii)* is evident. Let us prove the converse.

Since (X,π_X) is birational to (Y,π_Y) and both of them are minimal, the number of singular fibres of π_X and π_Y is the same, equal to $2r$ for some non-negative integer r.

Assume that $r=0$, which means that X is an Hirzebruch surfaces $\mathbb{F}_m$ for some m and that $Y=\mathbb{F}_n$ for some n. Since $X(\mathbb{R})$ is isomorphic to $Y(\mathbb{R})$, we have $m \equiv n \mod 2$. It is easy to prove that $(X(\mathbb{R}),\pi)$ and $(Y(\mathbb{R}),\pi)$ are isomorphic, by taking elementary links at two imaginary distinct fibres (see for example [Mang06, Proof of Theorem 6.1]).

When $r>0$, already the fact that (X,π_X) is birational to (Y,π_Y) implies that $(X(\mathbb{R}),\pi_X)$ is isomorphic to $(Y(\mathbb{R}),\pi_Y)$ (Proposition 22). □

5. Conic bundles on del Pezzo surfaces

In the preceding section we studied real conic bundles structures. In this section, we focus on surfaces admitting distinct minimal conic bundles. We will see that these surfaces are necessarily del Pezzo surfaces (Lemma 27). We begin by the description of all possible minimal real conic bundles occurring on del Pezzo surfaces.

Lemma 25. *Let V be is a subset of $\mathbb{P}^1(\mathbb{R})$, then the following are equivalent:*

i) *there exists a minimal real conic bundle (X,π) with $I(X,\pi)=V$ such that X is a del Pezzo surface;*
ii) *the set V is a union of closed intervals, and $\#V \leqslant 3$.*

Proof. The part *i)* $\Rightarrow$ *ii)* is easy. Indeed, if (X,π) is minimal, we know from Corollary 21 that its number of singular fibres is even, denoted $2r$, and that $2r = 8-(K_X)^2$. Since $-K_X$ is ample, $K_X^2 \geqslant 1$, thus $r \leqslant 3$. We conclude by applying once again Corollary 21 which asserts that $I(X,\pi)$ is the union of r closed intervals.

Let us prove the converse. If $V=\mathbb{P}^1(\mathbb{R})$ or $V=\emptyset$, we take (X,π) to be $(\mathbb{P}^1_\mathbb{C}\times\mathbb{P}^1_\mathbb{C}, \mathrm{pr}_1)$, where pr_1 is the projection on the first factor, endowed with the anti-holomorphic map that sends $\big((x_1:x_2),(y_1:y_2)\big)$ onto $\big((\overline{x_1}:\overline{x_2}),(\pm\overline{y_2}:\overline{y_1})\big)$.

If V consists of one interval, we consider the real surface given by

$$X := \left\{ ((x:y:z:t),(a:b)) \in \mathbb{P}^3_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}} \mid x^2+y^2+z^2=t^2, xb=ta \right\}.$$

The projection on $\mathbb{P}^3$ is a birational morphism from X to the smooth quadric $X_0 \subset \mathbb{P}^3$ given by $x^2+y^2+z^2=t^2$. Denote by by S_0 the complex surface obtained by forgetting the real structure. On S_0, we define two morphisms $\pi_{0,1}, \pi_{0,2} \colon S_0 \to \mathbb{P}^1_{\mathbb{C}}$; the map $\pi_{0,1}$ maps $(x:y:z:t)$ onto $(x-\mathbf{i}y : t-z)$ or $(t+z : x+\mathbf{i}y)$ and $\pi_{0,2}$ maps $(x:y:z:t)$ onto $(x-\mathbf{i}y : t+z)$ or $(t-z : x+\mathbf{i}y)$. The map $\pi_{0,1} \times \pi_{0,2}$ yields an isomorphism $S_0 \to \mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, so both S_0 and X_0 are del Pezzo surfaces of degree 8. The map $X \to X_0$ is the blow-up of the two imaginary points $(0:1:\pm\mathbf{i}:0)$. These two points, viewed on S_0, do not belong to the same fibre of $\pi_{0,1}$ or $\pi_{0,2}$ so X is a del Pezzo surface of degree 6. The map $\pi \colon X \to \mathbb{P}^1_{\mathbb{R}}$ can be taken as the second projection, up to an automorphism of $\mathbb{P}^1_{\mathbb{R}}$.

If V consists of two intervals, we consider the real surface given by

$$Y := \{(x:y:z:t) \in \mathbb{P}^3 \mid \lambda t(x^2+y^2+z^2-t^2)+2xyz=0\},$$

where $\lambda \in \mathbb{R}, \lambda > 1$. This is a smooth cubic surface, hence Y is a del Pezzo surface of degree 3. Denote by $\pi_Y \colon Y \to \mathbb{P}^1_{\mathbb{R}}$ the morphism which maps $(x:y:z:t)$ onto $(x:t)$ or $(\lambda(x^2+y^2+z^2-t^2) : -2yz)$. The fibre of $(1:0)$ is singular, and is the union of the two real lines of equation $y=t=0$ and $z=t=0$. The fibre of $(\alpha:1)$, $\alpha \in \mathbb{R}$ is the conic given by

$$\left\{ (\alpha t : y : z : t) \in \mathbb{P}^3 \;\middle|\; (y+\frac{\alpha}{\lambda}z)^2 + (1-\frac{\alpha^2}{\lambda^2})z^2 + (\alpha^2-1)t^2 = 0 \right\}.$$

This conic is singular if and only if $\alpha \in \{\pm 1, \pm\lambda\}$, and it has real points if and only if $\alpha \in]-\infty, -\lambda] \cup [-1,1] \cup [\lambda, \infty[$. Hence, $I(Y,\pi_Y)$ is the union of the two intervals $\{(x:1) \in \mathbb{P}^1_{\mathbb{R}} \mid -1 \leqslant x \leqslant 1\}$ and $\{(1:x) \in \mathbb{P}^1_{\mathbb{R}} \mid -1/\lambda \leqslant x \leqslant 1/\lambda\}$. Blowing-down the line $y=t=0$ (which is a (-1)-curve), we obtain a birational morphism $\eta \colon Y \to X$, where X is a del Pezzo surface of degree 4. Moreover, $\pi_X = \pi_Y \circ \eta^{-1}$ is a conic bundle with $I(X,\pi_X) = I(Y,\pi_Y)$. Choosing the right λ and up to an automorphism of $\mathbb{P}^1_{\mathbb{R}}$, we may choose that $I(X,\pi_X) = V$.

If V consists of three intervals I_1, I_2, I_3, we denote by m_1, m_2, m_3 three homogenous form of degree 2 such that m_j vanishes at the boundary of the interval I_j, and is non-negative on I_j. Define a surface

$$X := \left\{ ((x:y:z),(a:b)) \in \mathbb{P}^2_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}} \mid x^2 m_1(a,b) + y^2 m_2(a,b) + z^2 m_3(a,b) = 0 \right\}.$$

The projection on $\mathbb{P}^2_{\mathbb{R}}$ is a double covering. A straightforward calculation shows that this covering is ramified over a smooth quartic. In consequence, X is a smooth surface, and precisely a del Pezzo surface of degree 2. Taking $\pi \colon X \to \mathbb{P}^1_{\mathbb{R}}$ as the second projection, we get the result. □

Recall the following classical result, that will be useful in the sequel.

Lemma 26. *Let $\pi \colon S \to \mathbb{P}^1_{\mathbb{C}}$ be a complex conic bundle, and assume that S is a del Pezzo surface, with $(K_S)^2 = 9-m \leqslant 7$. Then, there exists a birational morphism $\eta \colon S \to \mathbb{P}^2_{\mathbb{C}}$ which is a blow-up of m points $p_1, \ldots, p_m$ and which sends the fibres of π onto the lines passing through p_1. The curves of self-intersection -1 of S are*

- *the exceptional curves $\eta^{-1}(p_1), \ldots, \eta^{-1}(p_m)$;*
- *the strict transforms of the lines passing through 2 of the p_i;*
- *the conics passing through 5 of the p_i;*
- *the cubics passing through 8 of the p_i and being singular at one of these.*

Proof. Denote by ε the contraction of one component in each singular fibre of π. Then, ε is a birational morphism of conic bundles – not defined over $\mathbb{R}$ – from S to a del Pezzo surface which is also an Hirzebruch surface. Changing the contracted components, we may assume that ε is a map $S \to \mathbb{F}_1$. Contracting the exceptional section onto a point $p_1 \in \mathbb{P}^2_{\mathbb{C}}$, we get a birational map $\eta\colon S \to \mathbb{P}^2_{\mathbb{C}}$ which is the blow-up of m points $p_1, \dots, p_m$ of $\mathbb{P}^2_{\mathbb{C}}$, and which sends the fibres of π_1 onto the lines passing through p_1. The description of the (-1)-curves is well-known and may be found for example in [Dem76]. □

Lemma 27. *Let $\pi_1\colon X \to \mathbb{P}^1_{\mathbb{R}}$ be a minimal real conic bundle. Then, the following conditions are equivalent:*

i) *There exist a real conic bundle $\pi_2\colon X \to \mathbb{P}^1_{\mathbb{R}}$, such that π_1 and π_2 induce distinct foliations on $X(\mathbb{C})$.*

ii) *Either X is isomorphic to $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$, or X is a del Pezzo surface of degree 2 or 4.*

Moreover, if the conditions are satisfied, then the following occur:

a) *The map π_2 is unique, up to an automorphism of $\mathbb{P}^1_{\mathbb{R}}$.*

b) *There exist $\alpha \in \mathrm{Aut}(X)$ and $\beta \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{R}})$ such that $\pi_1\alpha = \beta\pi_2$. Moreover, if X is a del Pezzo surface of degree 2, α may be chosen to be the Geiser involution.*

c) *Denoting by $f_1, f_2 \subset \mathrm{Pic}(X)$ the divisors of the general fibre of respectively π_1 and π_2, we have $f_1 + f_2 = -cK_X$ where $c = 4/(K_X)^2 \in \mathbb{N} \cdot \frac{1}{2}$.*

Proof. We now prove that *i)* implies *ii)*, *a)*, and *c)*. Assuming the existence of π_2, we denote by f_i the divisor of the fibre of π_i for $i = 1, 2$. We have $(f_1)^2 = (f_2)^2 = 0$ and by adjunction formula $f_1 \cdot K_X = f_2 \cdot K_X = -2$, where K_X is the canonical divisor. Let us write $d = (K_X)^2$.

Since (X, π_1) is minimal, $\mathrm{Pic}(X)$ has rank 2, hence $f_1 = aK_X + bf_2$, for some $a, b \in \mathbb{Q}$. Computing $(f_1)^2$ and $f_1 \cdot K_X$ we find respectively $0 = a^2d - 4ab = a(ad - 4b)$ and $-2 = ad - 2b$. If $a = 0$, we find $f_1 = f_2$, a contradiction. Thus, $4b = ad$ and $2b = ad + 2$, which yields $b = -1$ and $ad = -4$, so $f_1 + f_2 = -4/d \cdot K_X$. This shows that f_2 is uniquely determined by f_1, which is the assertion *a)*.

Denote as usual by S the complex surface associated to X. Let $C \in \mathrm{Pic}(S)$ be an effective divisor, with reduced support, and let us prove that $C \cdot (f_1 + f_2) > 0$. Since C is effective, $C \cdot f_1 \geqslant 0$ and $C \cdot f_2 \geqslant 0$. If $C \cdot f_1 = 0$, then the support of C is contained in one fibre of π_1. If C is a multiple of f_1, then $C \cdot f_2 > 0$; otherwise, C is a multiple of a (-1)-curve contained in a singular fibre of f_1, and the orbit of C by the anti-holomorphic involution is equal to a multiple of f_1, whence $C \cdot f_2 > 0$.

Since $f_1 + f_2$ is ample, and $f_1 + f_2 = -4/d \cdot K_X$ either K_X or $-K_X$ is ample. The surface X being geometrically rational, the former cannot occur, whence $d > 0$.

If S is isomorphic to $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$, the existence of π_1, π_2 shows that X is isomorphic to $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$. Otherwise, K_X is not a multiple in $\mathrm{Pic}(X_{\mathbb{C}})$ and thus d is equal to 1, 2 or 4. The number of singular fibres being even and equal to $8 - (K_X)^2$ by Corollary 21, the only possibilities are then 2 and 4.

We have proved that *i)* implies *ii)*, *a)*, and *c)*.

Assume now that $X = (S, \sigma)$ is $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$ or a del Pezzo surface of degree 2 or 4. We construct an automorphism α of X which does not belong to $\mathrm{Aut}(X, \pi)$. Then, by taking $\pi_2 = \pi_1\alpha$ we get assertion *i)*. Taking into account the unicity of π_2, we get *b)*.

If X is $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$, the two conic bundles are given by the projections on each factor, and we can get for α the swap of the factors.

If X is a del Pezzo surface of degree 2, the anti-canonical map $\zeta\colon X \to \mathbb{P}^2$ is a double covering ramified along a smooth quartic, cf. e.g. [Dem76]. Let α be the involution associated to the double covering – α is classically called the *Geiser involution*. It fixes a smooth quartic, hence cannot preserve any conic bundle.

The remaining case is when X is a del Pezzo surface of degree 4. By Lemma 26, there is a birational map $\eta\colon S \to \mathbb{P}^2_{\mathbb{C}}$ which is the blow-up of five points $p_1, \dots, p_5$ of $\mathbb{P}^2_{\mathbb{C}}$, no three being collinear and which sends the fibres of π_1 on the lines passing through p_1 . There are 16 exceptional curves (curves isomorphic to $\mathbb{P}^1_{\mathbb{C}}$ of self-intersection (-1)) on S:

- $E_1 = \eta^{-1}(p_1), ..., E_5 = \eta^{-1}(p_5)$ (5 curves);
- the strict transforms of the lines passing through p_i and p_j, denoted by L_{ij} (10 curves);
- the strict transform of the conic passing through the five points, denoted by Γ.

Note that the four singular fibres of π_1 are $E_i \cup L_{ij}$, $i = 2, \ldots, 5$, and that σ exchanges thus E_i and L_{ij} for $i = 1, \ldots, 5$. The intersection form being preserved, this implies that σ acts on the 16 exceptional curves as

$$(E_2\ L_{12})(E_3\ L_{13})(E_4\ L_{14})(E_5\ L_{15})(E_1\ \Gamma)(L_{23}\ L_{45})(L_{24}\ L_{35})(L_{25}\ L_{34}).$$

After a linear change of coordinates, we may assume that $p_1 = (1 : 1 : 1)$, $p_2 = (1 : 0 : 0)$, $p_3 = (0 : 1 : 0)$, $p_4 = (0 : 0 : 1)$ and $p_5 = (a : b : c)$ for some $a, b, c \in \mathbb{C}^*$. Denote by ϕ the birational involution $(x : y : z) \dashrightarrow (ayz : bxz : cxy)$ of $\mathbb{P}^2_{\mathbb{C}}$. Since the base-points of ϕ are p_2, p_3, p_4 and since ϕ exchanges p_1 and p_5, the map $\alpha = \eta^{-1}\phi\eta$ is an automorphism of S. Its action on the 16 exceptional curves is given by the permutation

$$(L_{23}\ E_4)(L_{24}\ E_3)(L_{34}\ E_2)(L_{12}\ L_{25})(L_{13}\ L_{35})(L_{14}\ L_{45})(\Gamma\ L_{15})(E_1\ E_5).$$

Observe that the actions of α and σ on the set of 16 exceptional curves commute. This means that $\alpha\sigma\alpha^{-1}\sigma^{-1}$ is an holomorphic automorphism of S which preserves any of the 16 curves. It is the lift of an automorphism of $\mathbb{P}^2_{\mathbb{C}}$ that fixes the 5 points $p_1, \ldots, p_5$ and hence is the identity. Consequently, α and σ commute, so $\alpha \in \mathrm{Aut}(X)$. Since ϕ sends a general line passing though p_1 onto a conic passing through $p_2, \ldots, p_5$, α belongs to $\mathrm{Aut}(X) \setminus \mathrm{Aut}(X, \pi)$. □

Corollary 28. *Let X be a minimal geometrically rational real surface, which is not rational. Then, the following are equivalent:*

i) $\#X(\mathbb{R}) = 2$ *or* $\#X(\mathbb{R}) = 3$*;*
ii) *There exists a geometrically rational real surface $Y(\mathbb{R})$ isomorphic to $X(\mathbb{R})$, and such that Y admits two minimal conic bundles $\pi_1 \colon Y \to \mathbb{P}^1_{\mathbb{R}}$ and $\pi_2 \colon Y \to \mathbb{P}^1_{\mathbb{R}}$ inducing distinct foliations on $Y(\mathbb{C})$.*

Proof. [*ii*) $\Rightarrow$ *i*)] By Lemma 27, Y is then a del Pezzo surface, which has degree 2 or 4 since Y is not rational. This implies that $\#Y(\mathbb{R}) = 2$ or $\#Y(\mathbb{R}) = 3$ by Proposition 12.

[*i*) $\Rightarrow$ *ii*)]. According to Theorem 10 and Proposition 12, (1) implies the existence of a minimal real conic bundle structure $\pi_X \colon X \to \mathbb{P}^1_{\mathbb{R}}$ with 4 or 6 singular fibres. This condition is equivalent to the fact that $I(X, \pi_X)$ is the union of 2 or 3 intervals (Corollary 21). According to Lemma 25, there exists a minimal real conic bundle (Y, π_1) such that Y is a del Pezzo surface and $I(Y, \pi_1) = I(X, \pi_X)$. Corollary 23 shows that $(X(\mathbb{R}), \pi_X)$ and (Y, π_1) are isomorphic. Moreover Lemma 27 yields the existence of π_2. □

6. Equivalence of surfaces versus equivalence of conic bundles

This section is devoted to the proof of Theorem 4. It remains to solve the conic bundle case, which is done in Theorem 31. First of all, we correct an existing inaccuracy in the literature; in [Kol97, Exercice 5.8] or [Sil89, VI.3.5], it is asserted that all minimal real conic bundles with four singular fibres belong to a unique birational equivalence class. To the contrary, the following general result, which includes the case with four singular fibres, occurs:

Theorem 29. *Let $\pi_X \colon X \to \mathbb{P}^1_{\mathbb{R}}$ and $\pi_Y \colon Y \to \mathbb{P}^1_{\mathbb{R}}$ be two real conic bundles, and suppose that either X or Y is non-rational. Then, the following are equivalent:*

i) *The two real surfaces X and Y are birational.*

ii) *The two real conic bundles (X,π_X) and (Y,π_Y) are birational.*

iii) *There exists an automorphism of $\mathbb{P}^1$ which sends $I(X,\pi_X)$ onto $I(Y,\pi_Y)$.*

Moreover, if the number of singular fibres of π_X is at least 8*, then* $\mathrm{Bir}(X)=\mathrm{Bir}(X,\pi_X)$.

Remark 30. It is well-known that this result is false when X and Y are rational. Indeed, consider $(X,\pi_X)=(\mathbb{P}^1_{\mathbb{R}}\times\mathbb{P}^1_{\mathbb{R}},\mathrm{pr}_1)$ and (Y,π_Y) be a real conic bundle with two singular fibres. The surfaces X and Y are birational, but the conic bundles (X,π_X) and (Y,π_Y) are not.

Proof. The equivalence $(iii)\Leftrightarrow(ii)$ was proved in Corollary 23 and $(ii)\Rightarrow(i)$ is evident.

Let us prove now $(i)\Leftrightarrow(ii)$. We may assume that (X,π_X) and (Y,π_Y) are minimal and that X is not rational, hence π_X has at least 4 singular fibres. Let $\psi\colon X\dashrightarrow Y$ a birational map, and decompose ψ into elementary links: $\psi=\psi_n\circ\cdots\circ\psi_1$ (see [Isk96, Theorem 2.5]). Consider $\psi_1\colon X\dashrightarrow X_1$ the first link, which may be of type II or IV only by [Isk96, Theorem 2.6]. If ψ_1 is of type II, then ψ_1 is a birational map of conic bundles $(X,\pi_X)\dashrightarrow(X_1,\pi_1)$ for some conic bundle structure $\pi_1\colon X_1\to\mathbb{P}^1$. If ψ_1 is of type IV, then ψ_1 is an isomorphism $X\to X_1$ and the link is precisely a change of conic bundle structure from π_X to $\pi_1\colon X_1\to\mathbb{P}^1$, which induce distinct foliations on $X(\mathbb{R})$. Applying Lemma 27, X is a del Pezzo surfaces of degree 2 or 4, and there exist automorphisms $\alpha\in\mathrm{Aut}(X)$ and $\beta\in\mathrm{Aut}(\mathbb{P}^1_{\mathbb{R}})$ such that $\pi_1\psi_1\alpha=\beta\pi_2$, whence (X,π) is isomorphic to (X_1,π_1). We proceed by induction on the number of elementary links to conclude that (X,π_X) is birational to (Y,π_Y). Moreover, if π_X has at least 8 singular fibres, then no link of type IV may occur, so ψ is a birational map of conic bundles $(X,\pi_X)\dashrightarrow(Y,\pi_Y)$. □

When the conic bundles are minimal, we can strengthen Theorem 29 to get an isomorphism between the real parts.

Theorem 31. *Let $\pi_X\colon X\to\mathbb{P}^1_{\mathbb{R}}$ and $\pi_Y\colon Y\to\mathbb{P}^1_{\mathbb{R}}$ be two* minimal *real conic bundles, and suppose that either X or Y is non-rational. Then, the following are equivalent:*

i) *X and Y are birational.*

ii) *$X(\mathbb{R})$ and $Y(\mathbb{R})$ are isomorphic.*

iii) *$(X(\mathbb{R}),\pi_X)$ and $(Y(\mathbb{R}),\pi_Y)$ are isomorphic.*

Proof. The implications iii) ⇒ ii) ⇒ i) being evident, it suffices to prove i) ⇒ iii). Since X and Y are not rational, both π_X and π_Y have at least one singular fibre. Applying Lemma 19, we may assume that both (X,π_X) and (Y,π_Y) are exceptional real conic bundles. Then, since (X,π_X) and (Y,π_Y) are birational (Theorem 29), we may assume that $I(X,\pi_X)=I(Y,\pi_Y)$, up to an automorphism of $\mathbb{P}^1_{\mathbb{R}}$. Then Corollary 23 shows that (X,π_X) is isomorphic to (Y,π_Y). □

We are now able to prove Theorem 4 concerning minimal surfaces.

Proof of Theorem 4. Let X and Y be two minimal geometrically rational real surfaces, and assume that either X or Y is non-rational.

If $X(\mathbb{R})$ and $Y(\mathbb{R})$ are isomorphic, it is clear that X and Y are birational. Let us prove the converse.

Theorem 10 lists all the possibilities for X. If $\rho(X)=1$ or $\rho(Y)=1$, Proposition 13 shows that X is isomorphic to Y. Otherwise, since neither X nor Y is rational, there exist minimal conic bundle structures on X and on Y. From Theorem 31, we conclude that $X(\mathbb{R})$ is isomorphic to $Y(\mathbb{R})$. □

To go further with non-minimal surfaces, we need to know when the group $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is very transitive for X minimal. This is done in the next sections.

7. Very transitive actions

Thanks to the work done in Section 4, it is easy to apply the techniques of [HM09a] to prove that $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is fiberwise very transitive on a real conic bundle. After describing the transitivity of $\mathrm{Aut}\big(X(\mathbb{R})\big)$ on the tangent space of a general point, we set the main result of that section: $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is very transitive on each connected component when X is minimal and admits two conic bundle structures (Proposition 36). We end the section by giving a characterisation of surfaces X for which $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is able to mix the connected components of $X(\mathbb{R})$.

Lemma 32. *Let (X,π) be a minimal real conic bundle over $\mathbb{P}^1_{\mathbb{R}}$ with at least one singular fibre. Let $(p_1,\dots,p_n)$ and $(q_1,\dots,q_n)$ be two n-tuples of distinct points of $X(\mathbb{R})$, and let $(b_1,\dots,b_m)$ be m points of $I(X,\pi)$. Assume that $\pi(p_i)=\pi(q_i)$ for each i, that $\pi(p_i)\neq\pi(p_j)$ for $i\neq j$ and that $\pi(p_i)\neq b_j$ for any i and any j.*

Then, there exists $\alpha\in\mathrm{Aut}\big(X(\mathbb{R})\big)$ such that $\alpha(p_i)=q_i$ for every i, $\pi\alpha=\pi$ and $\alpha|_{\pi^{-1}(b_i)}$ is the identity for every i.

Remark 33. The same result holds for minimal real conic bundles with no singular fibre, see [BH07, 5.4]. The following proof uses *twisting maps*, see below, which were introduced in [HM09a] to prove that the action of the group of automorphisms $\mathrm{Aut}(S^2)$ on the quadric sphere $S^2:=\{(x:y:z)\in\mathbb{R}^3\mid x^2+y^2+z^2=0\}$ is very transitive.

Proof. By Lemma 19, we may assume that (X,π) is exceptional. Moreover, Lemma 18 yields the existence of an affine real surface $A\subset X$ isomorphic to the hypersurface of $\mathbb{R}^3$ given by

$$y^2+z^2=-\prod_{i=1}^{2r}(x-a_i),$$

for some $a_1,\dots,a_{2r}\in\mathbb{R}$ with $a_1<a_2<\cdots<a_{2r}$, where $\pi|_A$ corresponds to the projection $(x,y,z)\mapsto x$, and where the inclusion $A\subset X$ induces an isomorphism $A(\mathbb{R})\to X(\mathbb{R})$.

For $i=1,\dots,n$, let us denote by (x_i,y_i,z_i) the coordinates of p_i in $A\subset\mathbb{R}^3$ and by (u_i,v_i,w_i) the ones of q_i. From hypothesis, we have $x_i=u_i$ for all i, thus we get $y_i^2+z_i^2=v_i^2+w_i^2$ for all i. Let $\Phi_i\in\mathrm{SO}_2(\mathbb{R})$ be the rotation sending (x_i,y_i) to (u_i,v_i). Then by [HM09a, Lemma 2.2], there exists an algebraic map $\Phi\colon[a_1,a_{2r}]\to\mathrm{SO}_2(\mathbb{R})$ such that $\Phi(x_i)=\Phi_i$ for $i=1,\dots,n$ and $\Phi(b_i)$ is the identity for $i=1,\dots,m$. Let us recall the proof; since $\mathrm{SO}_2(\mathbb{R})$ is isomorphic to the unit circle $S^1:=\{(x:y:z)\in\mathbb{P}^2(\mathbb{R})\mid x^2+y^2=z^2\}$, it suffices to prove the statement for S^1 instead of $\mathrm{SO}_2(\mathbb{R})$. Let Φ_0 be a point of S^1 distinct from $\Phi_1,\dots,\Phi_n$ and from the identity. Since $S^1\setminus\{\Phi_0\}$ is isomorphic to $\mathbb{R}$, it suffices, finally, to prove the statement for $\mathbb{R}$ instead of $\mathrm{SO}_2(\mathbb{R})$. The latter statement is an easy consequence of Lagrange polynomial interpolation.

Then the map defined by $\alpha\colon(x,y,z)\mapsto\big(x,(y,z)\cdot\Phi(x)\big)$ induces an automorphism $A(\mathbb{R})\to A(\mathbb{R})$ called the *twisting map* of π associated to Φ. Moreover, $\alpha(p_i)=q_i$, for all i, $\pi\alpha=\pi$, $\alpha|_{\pi^{-1}(b_i)}$ is the identity for every i, and π induces an automorphism $X(\mathbb{R})\to X(\mathbb{R})$. □

Lemma 34. *Let (X,π) be a minimal real conic bundle over $\mathbb{P}^1_{\mathbb{R}}$ with at least one singular fibre. Let $p\in X$ be a real point in a nonsingular fibre of π, and let $\Sigma\subset I(X,\pi)$ be a finite subset, with $\pi(p)\in\Sigma$. Denote by $\eta\colon Y\to X$ the blow-up of p, and by $E\subset Y$ the exceptional curve. Let $q\in E$ the point corresponding to the direction of the fibre of π passing through p.*

Then, the lift of the group

$$G=\Big\{\alpha\in\mathrm{Aut}\big(X(\mathbb{R})\big),\pi\alpha=\pi\;\Big|\;\alpha|_{\pi^{-1}(\Sigma)}\text{ is the identity}\Big\}$$

by η is a subgroup $\eta^{-1}G\eta\subset\mathrm{Aut}\big(Y(\mathbb{R})\big)$ which fixes the point q, and acts transitively on $E\backslash q\cong\mathbb{A}^1_{\mathbb{R}}$.

Proof. Since G acts identically on $\pi^{-1}(\Sigma)$, it fixes p, and therefore lifts to $H = \eta^{-1}G\eta \subset \mathrm{Aut}(Y(\mathbb{R}), \pi\eta)$, which preserves E. Moreover, G preserves the fibre of π passing through p, so H preserves its strict transform, which intersects transversally E at q, so q is fixed.

Let us prove now that the action of $\eta^{-1}G\eta$ on $E\backslash q$ is transitive. By Lemma 19, we may assume that (X, π) is exceptional. Then, we take an affine surface $A \subset X$, isomorphic to the hypersurface $y^2+z^2 = P(x)$ of $\mathbb{R}^3$ for some polynomial P, such that $A|_\pi$ is the projection $\mathrm{pr}_x\colon (x, y, z) \mapsto x$ and the inclusion $A \subset X$ gives an isomorphism $A(\mathbb{R}) \to X(\mathbb{R})$ (Lemma 18). Let us write $(x_0, y_0, z_0) \in \mathbb{R}^3$ the coordinates of p. Since x is on a nonsingular fibre of π, then $P(x_0) > 0$. Up to an affine automorphism of $\mathbb{R}^3$, and up to multiplication of P by some constant, we may assume that $x_0 = 0$, $P(0) = 1$, $y_0 = 0$, and $z_0 = 0$.

To any real polynomial $\lambda \in \mathbb{R}[X]$, we associate the matrix

$$\begin{pmatrix} \alpha(X) & \beta(X) \\ -\beta(X) & \alpha(X) \end{pmatrix} \in \mathrm{SO}_2(\mathbb{R}(X)) ,$$

where $\alpha = \frac{1-\lambda^2}{1+\lambda^2} \in \mathbb{R}(X)$ and $\beta = \frac{2\lambda}{1+\lambda^2} \in \mathbb{R}(X)$. And corresponding to this matrix, we associate the map

$$\psi_\lambda\colon (x, y, z) \mapsto (x, \alpha(x)\cdot y - \beta(x)\cdot z, \beta(x)\cdot y + \alpha(x)\cdot z),$$

which belongs to $\mathrm{Aut}(A(\mathbb{R}), \mathrm{pr}_x)$. To impose that ψ_λ is the identity on $(\mathrm{pr}_x)^{-1}(\Sigma)$ is the same to ask that $\lambda(x) = 0$ for each $(x : 1) \in \Sigma \subset \mathbb{P}^1(\mathbb{R})$, and in particular for $x = 0$.

Denote by $\mathcal{O} = \mathbb{R}[x, y, z]/(y^2 + z^2 - P(x))$ the ring of functions of A, by $\mathfrak{p} \subset \mathcal{O}$ the ideal of functions vanishing at p, by $\mathcal{O}_\mathfrak{p}$ the localisation, and by $\mathfrak{m} \subset \mathcal{O}_\mathfrak{p}$ the maximal ideal of $\mathcal{O}_\mathfrak{p}$. Then, the cotangent ring $T^*_{p,A}$ of p in A is equal to $\mathfrak{m}/\mathfrak{m}^2$, and is generated by the images $[x]$, $[y]$, $[z-1]$ of $x, y, z-1 \in \mathbb{R}[x, y, z]$. Since $P(0) = 1$, we may write $P(x) = 1 + xQ(x)$, for some real polynomial Q. We compute

$$[0] = [y^2 + z^2 - P(x)] = [y^2 + (z-1)^2 + 2(z-1) - xQ(x)] = [2(z-1) - xQ(0)] \in \mathfrak{m}/\mathfrak{m}^2 .$$

We see that $[z-1] = [xQ(0)/2]$, thus $\mathfrak{m}/\mathfrak{m}^2$ is generated by $[x]$ and $[y]$ as a $\mathbb{R}$-module. Since $\lambda(0) = 0$, we can write $\lambda(x) = x\mu(x)$, for some real polynomial μ. The linear action of ψ_λ on the cotangent space $T^*_{p,A}$ fixes $[x]$ and sends $[y]$ onto

$$\begin{array}{rcl} [\alpha(x)\cdot y - \beta(x)\cdot z] & = & \left[\frac{(1-\lambda(x)^2)y - 2\lambda(x)z}{\lambda(x)^2+1}\right] = [y - 2\lambda(x)(1 + xQ(0)/2)] \\ & = & [y - 2\mu(0)x] . \end{array}$$

It suffices to change the derivative of λ at 0 (which is equal to $\mu(0)$), which may be any real number. Therefore, the action of G on the projectivisation of $T^*_{p,A}$, fixes a point (corresponding to $[x]$) but acts transitively on the complement of this point. Since E corresponds to the projectivisation of $T_{p,A}$, G acts transitively on $E\backslash q$. □

Lemma 35. *Let X be a real projective surface endowed with two minimal conic bundles $\pi_1\colon X \to \mathbb{P}^1_\mathbb{R}$ and $\pi_2\colon X \to \mathbb{P}^1_\mathbb{R}$ inducing distinct foliations on $X(\mathbb{C})$. Let F_j be a real fibre of π_j, $j = 1, 2$. If $F_1(\mathbb{R}) \cap F_2(\mathbb{R}) \neq \emptyset$, then at most one of the curves F_j can be singular.*

Proof. Suppose the converse for contradiction. Then, F_i is the union of two (-1)-curves $E_{i,1}$ and $E_{i,2}$, intersecting transversally at some point p_i. Since p_i is the only real point of F_i, we have $p_1 = p_2$. Hence, $E_{2,1}\cdot F_1 \geqslant 2$.

According to Lemma 27, X is a del Pezzo surface of degree 2 or 4. Denote by S the complex surface obtained by forgetting the real structure on X, and by $\eta\colon S \to \mathbb{P}^2_\mathbb{C}$ the birational map which is the blow-up of $p_1, \ldots, p_m$, $m = 5$ or $m = 7$, and which sends the fibres of π_1 on lines passing through p_1 (Lemma 26). The curves $E_{2,1}$ and $E_{2,2}$ having self-intersection -1, these are the strict transform of lines or conics of $\mathbb{P}^2$ passing through 3 or 5 of the p_i. Since both curves intersect the fibres of π_1 into at least 2 points, the curves are conics not passing through p_1. This means that $m = 7$ and that the two conics intersect nowhere except at four of the points $p_2, \dots, p_7$. This is impossible since $E_{2,1}$ and $E_{2,2}$ intersect at p. $\square$

We now use the above lemmas to show that the action of $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is very transitive on each connected component when X is a surface with two conic bundles.

PROPOSITION 36. *Let X be a real projective surface, which admits two minimal conic bundles $\pi_1\colon X \to \mathbb{P}^1_{\mathbb{R}}$ and $\pi_2\colon X \to \mathbb{P}^1_{\mathbb{R}}$ inducing distinct foliations on $X(\mathbb{C})$.*

Let $(p_1, \dots, p_n)$ and $(q_1, \dots, q_n)$ be two n-tuples of distinct points of $X(\mathbb{R})$ such that p_i and q_i belong to the same connected component for each i. Then, there exists an element of $\mathrm{Aut}\big(X(\mathbb{R})\big)$ which sends p_i on q_i for each i, and which sends each connected component of $X(\mathbb{R})$ on itself.

Proof. When X is rational, the result follows from [HM09a, Theorem 1.4]. Thus we assume that X is non-rational, and in particular that $X(\mathbb{R})$ is non-connected.

From Lemma 35, any real point which is critical for one fibration is not critical for the second fibration. Otherwise speaking (recall that the fibrations are minimal) a real intersection point of a fibre F_1 of π_1 with a fibre F_2 of π_2 cannot be a singular point of F_1 and of F_2 at the same time. By Lemma 32 applied to (X, π_1), and to (X, π_2), we may assume without loss of generality that all points $p_1, \dots, p_n, q_1, \dots, q_n$ belong to smooth fibres of π_1 and to smooth fibres of π_2. We now use Lemma 32 to obtain an automorphism α of $(X(\mathbb{R}), \pi_1)$ such that $\pi_2(\alpha(p_i)) \neq \pi_2(\alpha(p_j))$ and $\pi_2(\alpha(q_i)) \neq \pi_2(\alpha(q_j))$ for $i \neq j$. Hence, we may suppose that $\pi_2(p_i) \neq \pi_2(p_j)$ and $\pi_2(q_i) \neq \pi_2(q_j)$ for $i \neq j$.

Likewise, using an automorphism of $(X(\mathbb{R}), \pi_2)$ we may suppose that $\pi_1(p_i) \neq \pi_1(p_j)$ and $\pi_1(q_i) \neq \pi_1(q_j)$ for $i \neq j$.

We now show that for $i = 1, \dots, m$, there exists an element $\alpha_i \in \mathrm{Aut}\big(X(\mathbb{R})\big)$ that sends p_i on q_i and that restricts to the identity on the sets $\cup_{j\neq i}\{p_j\}$ and $\cup_{j\neq i}\{q_j\}$. Then, the composition of the α_i will achieve the proof. Observe that $\zeta = \pi_1 \times \pi_2$ gives a finite surjective morphism $X \to \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$ which is 2-to-1 or 4-to-1 depending of the degree of X (follows from assertion (c) of Lemma 27). Denote by W the image of $X(\mathbb{R})$. The map $X(\mathbb{R}) \to W$ is a differential map, which has topological finite degree. Denote by W_i the connected component of W which contains both $\zeta(p_i)$ and $\zeta(q_i)$. Observe that W_i is contained in the square $I(X, \pi_1) \times I(X, \pi_2)$, and that for each point $x \in W_i$, the intersection of the horizontal and vertical lines (fibres of the two projections of $\mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$) passing through x with W_i is either only $\{x\}$, when x is on the boundary of W_i, or is a bounded interval. Moreover, W_i is connected. Then, there exists a path from $\zeta(p_i)$ to $\zeta(q_i)$ which is a sequence of vertical or horizontal segments contained in W_i. We may furthermore assume that none of the segments is contained in $(\mathrm{pr}_1)^{-1}(\pi_1(a))$ or $(\mathrm{pr}_2)^{-1}(\pi_2(a))$ for any $a \in (\cup_{j\neq i}\{p_j\}) \cup (\cup_{j\neq i}\{q_j\})$. Denote by $r_1, ..., r_l$ the points of U that are sent on the singular points or ending points of the path, and by $s_1, \dots, s_l$ some points of $X(\mathbb{R})$ which are sent by ζ on $r_1, \dots, r_l$ respectively. Up to renumbering, $s_1 = p_i, s_l = q_i$ and two consecutive points s_j and s_{j+1} are such that $\pi_1(s_j) = \pi_1(s_{j+1})$ or $\pi_2(s_j) = \pi_2(s_{j+1})$. We construct then α_i as a composition of $l-1$ maps, each one belonging either to $\mathrm{Aut}(X(\mathbb{R}), \pi_1)$ or $\mathrm{Aut}(X(\mathbb{R}), \pi_2)$ and sending s_j on s_{j+1}, and fixing the points $(\cup_{j\neq i}\{p_j\}) \cup (\cup_{j\neq i}\{q_j\})$. $\square$

The following proposition describes the possible mixes of connected components.

PROPOSITION 37. *Let (X, π) be a minimal real conic bundle. Denote by $I_1, \dots, I_r$ the r connected components of $I(X, \pi)$, and by $M_1, \dots, M_r$ the r connected components of $X(\mathbb{R})$, where $I_i = \pi(M_i)$, $M_i = \pi^{-1}(I_i) \cap X(\mathbb{R})$. If $\nu \in \mathrm{Sym}_r$ is a permutation of $\{1, \dots, r\}$, the following are equivalent:*

i) *there exists $\alpha \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{R}})$ such that $\alpha(I_i) = I_{\nu(i)}$ for each i;*

ii) *there exists* $\beta \in \mathrm{Aut}(X(\mathbb{R}),\pi)$ *such that* $\beta(M_i) = M_{\nu(i)}$ *for each* i*;*

iii) *there exists* $\beta \in \mathrm{Aut}\big(X(\mathbb{R})\big)$ *such that* $\beta(M_i) = M_{\nu(i)}$ *for each* i*;*

iv) *there exist two real Zariski open sets* $V, W \subset X$*, and* $\beta \in \mathrm{Bir}(X)$*, inducing an isomorphism* $V \to W$*, such that* $\beta(V(\mathbb{R}) \cap M_i) = W(\mathbb{R}) \cap M_{\nu(i)}$ *for each* i*.*

Moreover, the conditions are always satisfied when $r \leqslant 2$*, and are in general not satisfied when* $r \geqslant 3$*.*

Proof. The implications $(ii) \Rightarrow (i)$ and $(ii) \Rightarrow (iii) \Rightarrow (iv)$ are obvious.

The implication $(i) \Rightarrow (ii)$ is a direct consequence of Corollary 23–ii).

We prove now that if $r \leqslant 2$, Assertion (i) is always satisfied, hence all the conditions are equivalent (since all are true). When $r \leqslant 1$, take α to be the identity. When $r = 2$, we make a linear change of coordinates to the effect that $I_1 = \{(x:1) \mid 0 \leqslant x \leqslant 1\}$ and I_2 is bounded by $(1:0)$ and $(\lambda:1)$, for some $\lambda \in \mathbb{R}$, $\lambda > 1$ or $\lambda < 0$. Then, $\alpha\colon (x_1 : x_2) \mapsto (\lambda x_2 : x_1)$ is an involution which exchanges I_1 and I_2.

It remains to prove the implication $(iv) \Rightarrow (i)$ for $r \geqslant 3$. We decompose β into elementary links

$$X = X_0 \overset{\beta_1}{\dashrightarrow} X_1 \overset{\beta_2}{\dashrightarrow} \cdots \overset{\beta_{n-1}}{\dashrightarrow} X_{n-1} \overset{\beta_n}{\dashrightarrow} X_n = X$$

as in [Isk96, Theorem 2.5]. It follows from the description of the links of [Isk96, Theorem 2.6] that each of the links is of type II or IV, and that the links of type II are birational maps of conic bundles and the links of type IV occur on del Pezzo surfaces of degree 2.

In consequence, each of the X_i admits a conic bundle structure given by $\pi_i\colon X_i \to \mathbb{P}^1_{\mathbb{R}}$, where $\pi_0 = \pi_n = \pi$, and if β_i has type II, it is a birational map of conic bundles $(X_{i-1}, \pi_{i-1}) \dashrightarrow (X_i, \pi_i)$, and if it has type IV, it is an isomorphism $X_{i-1} \to X_i$ which does not send the general fibre of π_{i-1} on those of π_i. In this latter case, since π_i and $\pi_{i-1}\beta_i$ have distinct general fibres, X_{i-1} and X_i are del Pezzo surfaces of degree 2, and the Geiser involution $\iota_{i-1} \in \mathrm{Aut}(X_{i-1})$ exchanges the two general fibres (follows from [Isk96, Theorem 2.6], but also from Lemma 27). This means that the map $\beta_i \circ \iota_{i-1}$, that we denote by γ_i, is an isomorphism of conic bundles $(X_{i-1}, \pi_{i-1}) \to (X_i, \pi_i)$.

Now, we prove by induction on the number of links of type IV that β may be decomposed into compositions of elements of $\mathrm{Bir}(X,\pi)$ and maps of the form $\psi\iota\psi^{-1}$ where ψ is a birational map of conic bundles $(X,\pi) \dashrightarrow (X',\pi')$, (X',π') is a del Pezzo surface of degree 2 and $\iota \in \mathrm{Aut}(X')$ is the Geiser involution. If there is no link of type IV, β preserves the conic bundle structure given by π. Otherwise, denote by β_i the first link of type IV, which is an isomorphism $\beta_i\colon X_i \to X_{i+1}$, and write $\beta_i = \gamma_i \circ \iota_{i-1}$ as before. We write $\psi = \beta_{i-1} \circ \cdots \circ \beta_1$, which is a birational map of conic bundles $\psi\colon (X,\pi) \dashrightarrow (X_i, \pi_i)$. Then, $\beta = (\beta_n \circ \cdots \circ \beta_{i+1} \circ \gamma_i \circ \psi)(\psi^{-1}\iota_{i-1}\psi)$. Applying the induction hypothesis on the map $(\beta_n \circ \cdots \circ \beta_{i+1} \circ \gamma_i \circ \psi) \in \mathrm{Bir}(X)$, we are done.

Now, observe that when (X',π') is a minimal real conic bundle and X' is a del Pezzo surface of degree 2, the map $\zeta\colon X' \to \mathbb{P}^2_{\mathbb{R}}$ given by $|-K_{X'}|$ is a double covering, ramified over a smooth quartic curve $\Gamma \subset \mathbb{P}^2_{\mathbb{R}}$ (see e.g. [Dem76]). Since (X,π) is minimal, π has 6 singular fibres (Corollary 21), so $I(X,\pi)$ is the union of three intervals and $X(\mathbb{R})$ is the union of 3 connected components (Corollary 21). This implies that $\Gamma(\mathbb{R})$ is the union of three disjoint ovals. A connected component M of $X(\mathbb{R})$ is homeomorphic to a sphere, and surjects by ζ to the interior of one of the three ovals. The Geiser involution (induced by the double covering) induces an involution on M, which fixes the preimage of the oval. This means that the Geiser involution sends any connected component of $X(\mathbb{R})$ on itself. Thus, in the decomposition of β into elements of $\mathrm{Bir}(X,\pi)$ and conjugate elements of Geiser involutions, the only relevant elements are those of $\mathrm{Bir}(X,\pi)$. There exists thus $\beta' \in \mathrm{Bir}(X,\pi)$ which acts on the connected components of $X(\mathbb{R})$ in the same way as β. This shows that (iv) implies (i).

We finish by proving that (i) is false in general, when $r \geqslant 3$. This follows from the fact that if Σ is a general finite subset of $2r$ distinct points of $\mathbb{P}^1_{\mathbb{R}}$, the group $\{\alpha \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{R}}) \mid \alpha(\Sigma) = \Sigma\}$ is trivial. Supposing this fact true, we obtain the result by applying it to the $2r$ boundary points of $I(X,\pi)$. Let us prove the fact. The set of $2r$-tuples of $\mathbb{P}^1_{\mathbb{R}}$ is an open subset W of $(\mathbb{P}^1_{\mathbb{R}})^{2r}$. For any non-trivial permutation $\upsilon \in \mathrm{Sym}_{2r}$, we denote by $W_\upsilon \subset W$ the set of points $a = (a_1, \ldots, a_{2r}) \in W$ such that there exists $\alpha \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{R}})$ with $\alpha(a_i) = a_{\upsilon(i)}$ for each i. Let $a \in W_\upsilon$, and take two 4-tuples Σ_1, Σ_2 of a_i's with $\Sigma_1 \neq \Sigma_2$ and $\Sigma_2 = \upsilon(\Sigma_1)$ (this is possible since υ is non-trivial). Then, the cross-ratio of the a_i's in Σ_1 and in Σ_2 are the same. This implies a non-trivial condition on W. Consequently, W_υ is contained in a closed subset of W. Doing this for all non-trivial permutations υ, we obtain the result. □

8. Real algebraic models

The aim of this section is to go further with non-minimal surfaces with 2 or 3 connected components. We begin to show how *to separate* infinitely near points to the effect that any such a surface $Y(\mathbb{R})$ is isomorphic to a blow-up $B_{a_1,\ldots,a_m}X(\mathbb{R})$ where X is minimal and $a_1, \ldots, a_m$ are distinct proper points of $X(\mathbb{R})$. Then, we replace $X(\mathbb{R})$ by an isomorphic del Pezzo model (Corollary 28) and we use the fact that $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is very transitive on each connected component for such an X (Proposition 36) to prove that in many cases, if two birational surfaces Y and Z have homeomorphic real parts then $Y(\mathbb{R})$ and $Z(\mathbb{R})$ are isomorphic. As a corollary, we get that in any cases, $\mathrm{Aut}\big(Y(\mathbb{R})\big)$ is very transitive on each connected component.

Proposition 38. *Let X be a minimal geometrically rational real surface, with $\#X(\mathbb{R}) = 2$ or $\#X(\mathbb{R}) = 3$, and let $\eta\colon Y \to X$ be a birational morphism.*

Then there exists a blow-up $\eta'\colon Y' \to X$, whose centre is a finite number of distinct real proper points of X, and such that $Y'(\mathbb{R})$ is isomorphic to $Y(\mathbb{R})$.

Moreover, we can assume that the isomorphism $Y(\mathbb{R}) \to Y'(\mathbb{R})$ induces an homeomorphism $\eta^{-1}(M) \to (\eta')^{-1}(M)$ for each connected component M of $X(\mathbb{R})$.

Proof. According to Corollary 28, we may assume that X admits two minimal conic bundles $\pi_1\colon X \to \mathbb{P}^1_{\mathbb{R}}$ and $\pi_2\colon X \to \mathbb{P}^1_{\mathbb{R}}$ inducing distinct foliations on $X(\mathbb{C})$. Preserving the isomorphism class of $Y(\mathbb{R})$, we may assume that the points in the centre of η are all real (such a point may be a proper point of $X(\mathbb{R})$ or an infinitely near point). Let us denote by m $(= K_X^2 - K_Y^2)$ the number of those points. We prove the result by induction on m.

The cases $m = 0$ and $m = 1$ being obvious (take $\eta' = \eta$), we assume that $m \geqslant 2$. We decompose η as $\eta = \theta \circ \varepsilon$, where $\varepsilon\colon Y \to Z$ is the blow-up of one real point $q \in Z$, and $\theta\colon Z \to Y$ is the blow-up of $m-1$ real points. By induction hypothesis, we may assume that θ is the blow-up of $m-1$ proper points of X, namely $a_1, \cdots, a_{m-1} \in X(\mathbb{R})$. Moreover, applying Proposition 36, we may move the points by an element of $\mathrm{Aut}\big(X(\mathbb{R})\big)$, and assume that $\pi_1(a_i) \neq \pi_1(a_j)$ and $\pi_2(a_i) \neq \pi_2(a_j)$ for $i \neq j$, and that the fibre of π_1 passing through a_i and the fibre of π_2 passing through a_i are nonsingular and transverse at a_i, for each i.

If $\theta(q) \notin \{a_1, \ldots, a_{m-1}\}$, then η is the blow-up of m distinct proper points of X, hence we are done. Otherwise, assume that $\theta(q) = a_1$. We write $E = \theta^{-1}(a_1) \subset Z$, and denote by $F_i \subset Z$ the strict pull-back by η of the fibre of π_i passing through a_1, for $i = 1, 2$. Then, F_1 and F_2 are two (-1)-curves which do not intersect. Hence, the point $q \in E$ belongs to at most one of the two curves, so we may assume that $q \notin F_1$. Denote by $\theta_2\colon Z \to X_2$ the contraction of the $m-1$ disjoint (-1)-curves $F_1, \theta^{-1}(a_2), \ldots, \theta^{-1}(a_{m-1})$. Since q does not belong to any of these curves, $\eta_2 = \theta_2 \circ \varepsilon$ is the blow-up of $m-1$ distinct proper points of X_2. It remains to find an isomorphism $\gamma\colon X_2(\mathbb{R}) \to X(\mathbb{R})$ such that for each connected component M of $X(\mathbb{R})$, $\gamma\eta_2$ sends $\eta^{-1}(M)$ on M.

Denoting $\pi' = \pi_1 \circ \theta \circ \theta_2^{-1}$, the map $\psi = \theta_2 \circ \theta^{-1}$ is a birational map of conic bundles $(X, \pi_1) \dashrightarrow (X_2, \pi')$, which factorizes as the blow-up of a_1, followed by the contraction of the strict transform of the fibre passing through a_1. Therefore, the conic bundle (X_2, π') is minimal. Since $\#X(\mathbb{R}) > 1$ and $\pi'\psi = \pi_1$, Proposition 22 yields the existence of an isomorphism $\gamma\colon X_2(\mathbb{R}) \to X(\mathbb{R})$ such that $\pi_1\gamma = \pi'$. Observe that $\gamma\eta_2 \circ \eta^{-1} = \gamma\theta_2 \circ \theta^{-1} = \gamma\psi$ is a birational map $X \dashrightarrow X$ which satisfies $\pi \circ (\gamma\eta_2 \circ \eta^{-1}) = \pi$. Consequently, for any connected component M of $X(\mathbb{R})$, which corresponds to $\pi^{-1}(V) \cap X(\mathbb{R})$, for some interval $V \subset \mathbb{P}^1_{\mathbb{R}}$, we find $\pi(\gamma\eta_2\eta^{-1}(M)) = \pi(M) = V$, thus $\gamma\eta_2$ sends $\eta^{-1}(M)$ on M. □

Corollary 39. *Let X be a minimal geometrically rational real surface, such that $\#X(\mathbb{R}) = 2$ or $\#X(\mathbb{R}) = 3$, and let $\eta\colon Y \to X$, $\varepsilon\colon Z \to X$ be two birational morphisms. Denote by $M_1, \dots, M_r$ the connected components of $X(\mathbb{R})$ $(r = 2, 3)$. Then, the following are equivalent:*

i) *$\eta^{-1}(M_i) \subset Y(\mathbb{R})$ and $\varepsilon^{-1}(M_i) \subset Z(\mathbb{R})$ are homeomorphic for each i;*
ii) *there exists an isomorphism $Y(\mathbb{R}) \to Z(\mathbb{R})$ which induces an homeomorphism $\eta^{-1}(M_i) \to \varepsilon^{-1}(M_i)$ for each i.*

Proof. (2) $\Rightarrow$ (1) being obvious, let us prove the converse. According to Proposition 38, we may assume that η and ε are the blow-ups of a finite number of distinct real proper points of X. Denote by Σ_η and Σ_ε these two finite sets. For each i, the fact that $\eta^{-1}(M_i) \subset Y(\mathbb{R})$ and $\varepsilon^{-1}(M_i) \subset Z(\mathbb{R})$ are homeomorphic implies that the numbers of points of $\Sigma_\eta \cap M_i$ and $\Sigma_\varepsilon \cap M_i$ coincide.

By Corollary 28 and Proposition 36, $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is very transitive on each connected component of $X(\mathbb{R})$. In particular, there exists an element $\alpha \in \mathrm{Aut}\big(X(\mathbb{R})\big)$ such that $\alpha(M_i) = M_i$ for each i and $\alpha(\Sigma_\eta) = \Sigma_\varepsilon$. Then, $\psi = \varepsilon^{-1}\alpha\eta\colon Y(\mathbb{R}) \to Z(\mathbb{R})$ is the wanted isomorphism. □

Corollary 40. *Let Y be a geometrically rational real surface with $\#Y(\mathbb{R}) = 2$ or $\#Y(\mathbb{R}) = 3$. Let $(p_1, \dots, p_n)$ and $(q_1, \dots, q_n)$ be two n-tuples of distinct points of $Y(\mathbb{R})$ such that p_i and q_i belong to the same connected component for each i.*

Then, there exists an element $\alpha \in \mathrm{Aut}\big(Y(\mathbb{R})\big)$, which leaves each connected component of $Y(\mathbb{R})$ invariant and such that $\alpha(p_i) = q_i$ for each i.

Proof. Let $\eta\colon Y \to X$ be a birational morphism to a minimal real surface X; observe that $\#X(\mathbb{R}) = \#Y(\mathbb{R})$. According to Corollary 28, we may assume that X admits two minimal conic bundles $\pi_1\colon X \to \mathbb{P}^1_{\mathbb{R}}$ and $\pi_2\colon X \to \mathbb{P}^1_{\mathbb{R}}$ inducing distinct foliations on $X(\mathbb{C})$. By Proposition 38, we can suppose that η is the blow-up of m distinct real proper points $a_1, \dots, a_m \in X$. We prove the result by induction on m.

If $m = 0$, which means that $X = Y$, the result follows from Proposition 36.

If $m > 0$, denote by $\eta_0\colon Z \to X$ the blow-up of $a_1, \dots, a_{m-1}$ (η_0 is the identity if $m = 1$), and by $\eta_1\colon Y \to Z$ the blow-up of $b = \eta_0^{-1}(a_r)$.

Applying Proposition 36, we may assume that $\pi_1(a_i) \neq \pi_1(a_j)$ and $\pi_2(a_i) \neq \pi_2(a_j)$ for $i \neq j$, and that the fibre of π_1 passing through a_i and the fibre of π_2 passing through a_i are nonsingular and transverse at a_i, for each i. Let us denote by $E \subset Y$ the exceptional curve $\eta_1^{-1}(b)$ of η_1 and by F_i the strict transform on Y of the fibre of π_i passing through a_m, for $i = 1, 2$. Then E, F_1 and F_2 are three (-1)-curves, F_1 and F_2 do not intersect, and E intersect transversally each of the F_i. By induction hypothesis, we may use the lift of an element of $\mathrm{Aut}\big(Z(\mathbb{R})\big)$ which fixes b to assume that no one of the points p_i belongs to $F_1 \backslash E$, $F_2 \backslash E$ or to $\eta^{-1}(a_i)$ for $i = 1, \dots, m-1$. Then the group $G = \{\alpha \in \mathrm{Aut}\big(X(\mathbb{R})\big) \mid \pi_1\alpha = \pi_1, \alpha \text{ fixes } a_1, \dots, a_m, \eta(p_1), \dots, \eta(p_n)\}$, acts transitively on $E \backslash F_1$ (Lemma 34). Lifting a well-chosen element of this group in $\mathrm{Aut}\big(Y(\mathbb{R})\big)$, we may move the points p_i and assume that no one of the p_i belongs to F_2 (i.e. we can avoid $F_2 \cap E$). Denote by $\eta'\colon Y \to X'$ the contraction of the disjoint (-1)-curves $F_2, \eta^{-1}(a_1), \dots \eta^{-1}(a_{m-1})$.

Then, the birational map $\psi = \eta'\eta^{-1}\colon X \dashrightarrow X'$ is a birational map of conic bundles $(X, \pi_2) \dashrightarrow (X', \pi')$, where $\pi' = \pi_2\psi^{-1}$, which consists of the blow-up of a_m, followed by the contraction of the strict transform of the fibre passing through a_m. Therefore, the conic bundle (X', π') is minimal. Since $\#X(\mathbb{R}) > 1$, Proposition 22 yields the existence of an isomorphism $\gamma\colon X'(\mathbb{R}) \to X(\mathbb{R})$ such that $\pi_2\gamma = \pi'$. Therefore, there exists an element $\beta \in \mathrm{Aut}\big(X'(\mathbb{R})\big)$ which fixes all the points blown-up by η', which fixes all the points $\{\eta'(p_i), p_i \notin E\}$, and which sends the points $\{\eta'(p_i), p_i \in E\}$ outside of $\eta'(E)$. Applying the lift of β on $\mathrm{Aut}\big(Y(\mathbb{R})\big)$, we may assume that none of the points p_i belongs to E. Doing the same manipulation with the q_i, it remains to use the lift of an element of $\mathrm{Aut}\big(Z(\mathbb{R})\big)$ which fixes b and sends $\eta_1(p_i)$ on $\eta_1(q_i)$ for each i. □

9. Proof of the main results

The proof of Theorem 4 was given at the end of Section 5. Now, we deduce the others results stated in the introduction from the results of Sections 7 and 8. The following lemma serves to prove most of them.

Lemma 41. *Let (X, π) be a minimal real conic bundle, such that $I(X, \pi)$ is the union of r intervals $I_1, \dots, I_r$, with $r = 2$ or $r = 3$.*

Let $\eta_Y\colon Y \to X$ and $\eta_Z\colon Z \to X$ be two birational morphisms. For $i = 1, \dots, r$, we write $X_i = \pi^{-1}(I_i) \cap X(\mathbb{R})$, $Y_i = \eta_Y^{-1}(X_i) \cap Y(\mathbb{R})$ and $Z_i = \eta_Z^{-1}(X_i) \cap Z(\mathbb{R})$.

Let $p_1, \dots, p_n \in Y(\mathbb{R})$, $q_1, \dots, q_n \in Z(\mathbb{R})$ be two n-tuples of distinct points, and assume the existence of an homeomorphism $h\colon Y(\mathbb{R}) \to Z(\mathbb{R})$ which sends p_i on q_i for each i, and sends Y_i on $Z_{\nu(i)}$, where $\nu \in \mathrm{Sym}_r$ is a permutation of $\{1, \dots, r\}$. Then, the following are equivalent:

i) *There exists an isomorphism $\beta\colon Y(\mathbb{R}) \to Z(\mathbb{R})$ which sends Y_i on $Z_{\nu(i)}$ for each $i \in \{1, \dots, r\}$ and sends p_j on q_j for each $j \in \{1, \dots, n\}$.*

ii) *There exists an automorphism $\alpha \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{R}})$ which sends I_i on $I_{\nu(i)}$ for each $i \in \{1, \dots, r\}$.*

Moreover, both assertions are true if $r = 2$, and false in general when $r = 3$.

Proof. Observe that the X_i (respectively the Y_i, Z_i) are the connected components of $X(\mathbb{R})$ (respectively of $Y(\mathbb{R})$, $Z(\mathbb{R})$).

[$i) \Rightarrow ii)$] The map $\eta_Z\beta\eta_Y^{-1}$ is a birational self-map of X, which restricts to an isomorphism $\varphi\colon V \to W$, where V and W are two real Zariski open subsets of X. Moreover, the hypothesis on β implies that $\varphi(V(\mathbb{R}) \cap X_i) = W(\mathbb{R}) \cap X_{\nu(i)}$. The existence of α is provided by Proposition 37.

[$ii) \Rightarrow i)$] Proposition 37 yields the existence of $\gamma \in \mathrm{Aut}(X(\mathbb{R}), \pi)$ such that $\gamma(X_i) = X_{\nu(i)}$. We may thus assume that ν is the identity. According to Proposition 38, we may moreover suppose that η_Y and η_Z are the blow-ups of a finite set of disjoint real proper points of X. Since Y_i is homeomorphic to Z_i for each i, η_Y is the blow-up of $a_1, \dots, a_m$ and η_Z is the blow-up of $b_1, \dots, b_m$, where a_j and b_j belong to the same connected component of $X(\mathbb{R})$ for each j. Then, there exists an element of $\mathrm{Aut}\big(X(\mathbb{R})\big)$ which preserves each connected component of X and sends a_j on b_j for each j (Corollary 40). We may thus assume that $Y = Z$, and conclude by applying Corollary 40 to Y.

The fact that $ii)$ is true when $r = 2$ and false in general when $r = 3$ was proved in Proposition 37. □

The following case shares many features with the rational case.

Theorem 42. *Let X be a nonsingular geometrically rational real projective surface, and assume that $\#X(\mathbb{R}) = 2$. Then the action of the group $\mathrm{Aut}\big(X(\mathbb{R})\big)$ on $X(\mathbb{R})$ is very transitive.*

Proof of Theorem 42. Let Y be a nonsingular geometrically rational real projective surface, with $\#Y(\mathbb{R}) = 2$. Let $(p_1, \ldots, p_n)$ and $(q_1, \ldots, q_n)$ be two n-tuples of points which are compatible. We want to prove the existence of $\alpha \in \mathrm{Aut}\big(Y(\mathbb{R})\big)$ such that $\alpha(p_i) = q_i$ for each i.

If p_i and q_i are in the same connected component of $Y(\mathbb{R})$, the result follows from Corollary 40.

Otherwise, the compatibility means that the two components of $X(\mathbb{R})$ are homeomorphic and that p_i and q_i are in a distinct component for each i. Lemma 41 provides the existence of an element of $\mathrm{Aut}\big(Y(\mathbb{R})\big)$ which permutes the two connected components of $Y(\mathbb{R})$. This reduces the situation to the previous case. □

Theorem 2 is Corollary 40 applied to the case of 3 connected components.

Proof of Theorem 1. We prove firstly that if X is not geometrically rational, then $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is not very transitive and not 2-transitive. If X has Kodaira dimension 2, (surface of general type), it has only finitely many birational self-maps (see e.g. [Uen75].) If X has Kodaira dimension 1, every birational self-map of X preserves the elliptic fibration induced by $|K_X|$. If X has Kodaira dimension 0, and X is minimal, then $\mathrm{Bir}(X) = \mathrm{Aut}(X)$. The group $\mathrm{Aut}(X)$ is an algebraic group of dimension 1 or 2 (its neutral component is an elliptic curve or an Abelian surface). Thus, $\mathrm{Bir}(X)$ can not be 2-transitive. The case when X is not minimal is deduced from this case.

If X is a surface with Kodaira dimension $-\infty$, then X is uniruled. If furthermore, X is not geometrically rational and $X(\mathbb{R})$ is non-empty, then the Albanese map $X \to C$ is a real ruling over a curve with genus $g(C) > 0$, see e.g. [Sil89, V.(1.8)], and the Albanese map is preserved by any birational self-map.

Assume from now on that X is a geometrically rational surface. When $\#X(\mathbb{R}) = 1$, X is rational; the fact that $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is n-transitive for every n (and thus very transitive) is the main result of [HM09a]. When $\#X(\mathbb{R}) = 2$, $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is very transitive by Theorem 42.

When $\#X(\mathbb{R}) > 3$, we prove that the group $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is not transitive. Denote by $\eta\colon X \to X_0$ a birational morphism to a minimal real surface, and observe that $\#X_0(\mathbb{R}) = \#X(\mathbb{R}) > 3$. Let us discuss the two cases for X_0 given by Theorem 10. If X_0 is a del Pezzo surface with $\rho(X_0) = 1$, then $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is countable (Corollary 14), thus $\mathrm{Aut}\big(X(\mathbb{R})\big)$ cannot be transitive. The other case is when $\rho(X_0) = 2$. Then, X_0 endows a real conic bundle structure (X_0, π_0), and $\mathrm{Bir}(X_0) = \mathrm{Bir}(X_0, \pi_0)$ (Theorem 29). Since the action of $\mathrm{Bir}(X_0, \pi_0)$ on the basis of the conic bundle is finite (there are too much boundary points), neither $\mathrm{Aut}\big(X_0(\mathbb{R})\big)$ nor $\mathrm{Aut}\big(X(\mathbb{R})\big)$ may be transitive.

When $\#X(\mathbb{R}) = 3$, $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is very transitive on each connected component (Theorem 2). Thus, $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is very transitive if and only if for any homeomorphism $h\colon X(\mathbb{R}) \to X(\mathbb{R})$, there exists $\beta \in \mathrm{Aut}\big(X(\mathbb{R})\big)$ which permutes the components of $X(\mathbb{R})$ in the same way that h does. When these conditions are not satisfied, $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is not 2-transitive.

Let $X(\mathbb{R}) = M_1 \sqcup M_2 \sqcup M_3$ be the decomposition into connected components. If there is no pair (i, j) such that $M_i \sim M_j$, then there is no nontrivial such h, hence $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is very transitive. If $M_1 \sim M_2 \not\sim M_3$ or $M_1 \sim M_2 \sim M_3$, the possibilities when this occur follow from Lemma 41.

For example, when X is minimal (therefore $M_1 \sim M_2 \sim M_3 \sim S^2$), it admits a minimal real conic bundle structure (X, π) (Theorem 10 and Proposition 12), where π has 6 singular fibres. Then, $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is very transitive if and only if $\{\alpha \in \mathrm{Aut}(\mathbb{P}^1_{\mathbb{R}}) \mid \alpha(I(X, \pi)) = I(X, \pi)\}$ acts transitively on the three intervals of $I(X, \pi)$. This is true in some special cases, but false in general. When X is not minimal, $\mathrm{Aut}\big(X(\mathbb{R})\big)$ is very transitive for example when there is no pair of homeomorphic connected components of $X(\mathbb{R})$, or when X is the blow-up of a minimal surface Y with a very transitive group $\mathrm{Aut}\big(Y(\mathbb{R})\big)$. □

Proof of Theorem 3. Let X, Y be two geometrically rational real surfaces, and assume that $\#X(\mathbb{R}) \leqslant 2$. We assume that X is birational to Y and that $X(\mathbb{R})$ is homeomorphic to $Y(\mathbb{R})$, and prove that

$X(\mathbb{R})$ is isomorphic to $Y(\mathbb{R})$.

Remark that all geometrically rational surfaces with connected real part are birational to each others, thus in this case the statement follows from the unicity of rational models [BH07]. We may thus assume that $\#X(\mathbb{R}) = 2$. Denote by $\eta_X : X \to X_0$ and $\eta_Y : Y \to Y_0$ birational morphisms to minimal real surfaces.

Since X_0 and Y_0 are birational, $X_0(\mathbb{R})$ and $Y_0(\mathbb{R})$ are isomorphic (Theorem 4), so we may assume that $X_0 = Y_0$. The result now follows from Lemma 41. □

Proof of Corollary 5*.* If M is connected, and M is non-orientable or M is orientable with genus $g(M) \leqslant 1$, then it admits a unique geometrically rational model by [BH07, Corollary 8.1]. Moreover, this model is in fact rational.

Conversely let M be a compact $\mathcal{C}^\infty$-surface and assume that M admits a unique geometrically rational model X. The existence of such a model implies, by Comessatti's theorem [Com14], that any connected component of M is non-orientable or is orientable with genus $g \leqslant 1$. The unicity means that for any geometrically rational model Y of M, then $Y(\mathbb{R})$ is isomorphic to $X(\mathbb{R})$. In particular, this implies that all geometrically rational models of M belong to a unique birational class. From Theorem 29 and Proposition 13, this means that X is rational. It remains to observe that when X is rational, $X(\mathbb{R})$ is connected, and is either non-orientable or orientable of genus $\leqslant 1$. When X is minimal, this follows from Proposition 12. Then, blowing-up points on a surface either does nothing on the topology of the real part (if the points blown-up are imaginary), or it gives a non-orientable real part (if the points blown-up ar real). □

We finish by a result on non-density. In [KM09], it is proved that $\operatorname{Aut}\big(X(\mathbb{R})\big)$ is dense in $\operatorname{Diff}\big(X(\mathbb{R})\big)$ when X is a geometrically rational surface with $\#X(\mathbb{R}) = 1$ (or equivalently when X is rational). In the cited paper, it is said that $\#X(\mathbb{R}) = 2$ is probably the only other case where the density holds. The following collect the known results in this direction. The first two of them are new.

Proposition 43. *Let X be a geometrically rational surface.*

- *If $\#X(\mathbb{R}) \geqslant 5$, then $\operatorname{Aut}\big(X(\mathbb{R})\big)$ is not dense in $\operatorname{Diff}\big(X(\mathbb{R})\big)$;*
- *if $\#X(\mathbb{R}) = 3$ or $\#X(\mathbb{R}) = 4$, then $\operatorname{Aut}\big(X(\mathbb{R})\big)$ is not dense in $\operatorname{Diff}\big(X(\mathbb{R})\big)$ for a general X, but could be dense in some special cases;*
- *if $\#X(\mathbb{R}) = 1$, then $\operatorname{Aut}\big(X(\mathbb{R})\big)$ is dense in $\operatorname{Diff}\big(X(\mathbb{R})\big)$.*

Proof. The case $\#X(\mathbb{R}) = 1$ is the main result of [KM09]. Assume from now on that $\#X(\mathbb{R}) \geqslant 3$, and denote by $\eta : X \to X_0$ a birational morphism to a minimal real surface, and observe that $\#X_0(\mathbb{R}) = \#X(\mathbb{R}) \geqslant 3$. Let us discuss the two cases for X_0 given by Theorem 10.

Assume that X_0 is a del Pezzo surface with $\rho(X_0) = 1$. If the degree of X_0 is 1 then $\operatorname{Bir}(X_0)$ is finite (Corollary 14), thus $\operatorname{Aut}\big(X(\mathbb{R})\big)$ cannot be dense. If X_0 has degree 2, then $\#X_0(\mathbb{R}) = 4$ (Proposition 12), so $\#X(\mathbb{R}) = 4$ too. Since $\operatorname{Aut}\big(X_0(\mathbb{R})\big) = \operatorname{Aut}(X_0)$ is finite, $\operatorname{Aut}\big(X_0(\mathbb{R})\big)$ cannot be dense (but maybe $\operatorname{Aut}\big(X(\mathbb{R})\big)$ could be).

The other case is when $\rho(X_0) = 2$. Then, X_0 endows a real conic bundle structure (X_0, π_0). If $\#X(\mathbb{R}) = \#X_0(\mathbb{R}) \geqslant 4$, then $\operatorname{Bir}(X_0) = \operatorname{Bir}(X_0, \pi_0)$ (Theorem 29), so $\operatorname{Aut}\big(X(\mathbb{R})\big)$ is not dense. If $\#X_0(\mathbb{R}) = 3$, then in general $\operatorname{Aut}\big(X_0(\mathbb{R})\big)$ does not exchanges the connected component of $X_0(\mathbb{R})$. Consequently, $\operatorname{Aut}\big(X_0(\mathbb{R})\big)$ is not dense (but maybe $\operatorname{Aut}\big(X(\mathbb{R})\big)$ could be, if the connected components of $X(\mathbb{R})$ are not homeomorphic). □

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