

SOME REMARKS ABOUT CURVES ON K3 SURFACES

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ABSTRACT. The article studies the geometry of curves on $K3$ surfaces. Its aim is to give an alternative proof to the fact that on every projective $K3$ surface there are rational curves.

INTRODUCTION

In this note I wish to present my attempt at understanding the geometry of the curves on projective $K3$ surfaces. Actually I shall restrict my attention to the Picard general $K3$ surfaces, that is those whose Picard group has rank one. The reason is that these form a dense subset in the moduli space of projective $K3$ surfaces.

A classical result, due to Bogomolov and Mumford, states that on every projective $K3$ surface there are rational curves and 1-dimensional families of elliptic curves. Its proof has been sketched in [10], and has been treated in full detail by Xi Chen in [1]. Actually, in *loc. cit.* the author proves more, namely:

Theorem *For any integers $n \geq 3$ and $d > 0$, the linear system $|\mathcal{O}_S(d)|$ on a general $K3$ surface in $S \hookrightarrow \mathbb{P}^n$ contains a $(p(d) - \delta)$ -dimensional family of irreducible nodal curves, whose arithmetic genus equals $p(d) := d^2(n - 1) + 1$, and which posses precisely δ nodes.*

The common feature of these approaches is that one first studies degenerations of the general $K3$ surfaces to special ones, namely to the so-called trigonal surfaces. On these latter one explicitly produces singular curves, usually a connected curve with two smooth rational components. The second step consists in deforming the trigonal surface, together with (smoothings of) the curve, back to the starting $K3$ surface.

While this method is very elegant and short, it does not give much insight into the geometry of curves on a general $K3$ surface. A geometric approach has been considered by M. Green and Ph. Griffiths in [6, Part B], where they studied the geometry of the dual variety $S^* \hookrightarrow \mathbb{P}^n$ of S . Their guiding idea is that curves with nodes are obtained as hyperplane sections $H \cap S$, such that the corresponding hyperplane H has simple tangency points with S . Due to technical difficulties, they have been able to carry over the study only for $K3$'s in \mathbb{P}^n , with $n = 3, 4$, and for 2-sheeted coverings of the plane. Another study concerning the deformations of nodal curves on $K3$ surfaces is done in [11], but no existence result is proved there.

The idea in this note is similar to that of M. Green and Ph. Griffiths but the methods are different: we investigate the stability of the restriction of the tangent bundle of the $K3$ surface S to the curves on it. The consequence of the stability is that curves on $K3$ surfaces are rigid, that is they can not be deformed *within their conformal class*. Therefore, a deformation of a nodal curve inside S induces a non-trivial deformation of the conformal structure of the normalization of the curve. Combining this with S. Diaz's upper bound in [3] on the dimension of complete subvarieties of the Deligne-Mumford space M_g , we get an inductive process which produces curves on S whose geometric genus is strictly decreasing. Our main

result is stated in theorem 2.3: under some technical hypothesis (the assumption $(\star\star)$ in section 2), we reprove the theorem above up a a certain number of nodes, by investigating the geometry of the stable map spaces in the sense of Kontsevich-Manin.

1. A STABILITY RESULT

In this section we investigate the properties of the restriction of the tangent bundle of a $K3$ surface to curves. Since the tangent bundle of a $K3$ surface is stable, the results of Flenner and Mehta/Ramanathan (see [4, 8] or [7, chapter 7]) imply that its restriction to the general hypersurface section is stable.

What we wish to find are *effective bounds* for the degree, which guarantee the stability of the restriction, and also investigate the analogous issue for nodal curves. Observe that for a general $K3$ surface $S \xrightarrow{|\mathcal{O}_S(1)|} \mathbb{P}^n$, Flenner's bound implies that the restriction of T_S to the *general member* of $|\mathcal{O}_S(d)|$ is still stable for $d \geq 3(n-1)$. As we will see, this bound is quite weak. Our main tool is Bogomolov's effective restriction theorem (see [7, section 7.3]).

We make the following convention: for a connected nodal curve $\hat{C} \hookrightarrow S$, and a torsion free sheaf of rank one $\hat{Q} \rightarrow \hat{C}$, we define *the degree* $\deg_{\hat{C}} \hat{Q} := \chi(\hat{Q}) + p_a(\hat{C}) - 1$, where $p_a(\hat{C}) = h^1(\mathcal{O}_{\hat{C}})$ is the arithmetic genus of \hat{C} . We shall repeatedly use the following computational lemma.

Lemma 1.1. (i) *Let \hat{C} be a reduced, irreducible, nodal curve, and let $\nu : C \rightarrow \hat{C}$ be its normalization. Consider a rank 2 vector bundle $E \rightarrow \hat{C}$, and a quotient line bundle $q : \nu^*E \rightarrow Q$ of ν^*E . We denote by $\{x_{11}, x_{21}\}, \dots, \{x_{1\delta}, x_{2\delta}\} \subset C$ the pairs of points which are identified by ν , and let $0 \leq \delta' \leq \delta$ be the number of pairs $\{x_{1j}, x_{2j}\}$ for which $q_{x_{1j}}$ and $q_{x_{2j}}$ are not collinear. There is a torsion-free quotient $\hat{q} : E \rightarrow \hat{Q}$ of E , of degree $\deg_{\hat{C}} \hat{Q} = \deg_C Q + \delta'$, such that $Q = \nu^*\hat{Q}/(\nu^*\hat{Q})_{\text{tor}}$.*

(ii) *With the previous notations, assume that \hat{C} is a curve on a $K3$ -surface S , and that $E := T_S \otimes_{\mathcal{O}_S} \mathcal{O}_{\hat{C}}$. Denote by G the kernel of the composition $T_S \twoheadrightarrow E \twoheadrightarrow \hat{Q}$. Then G is a locally free, rank 2 sheaf on S , with*

$$c_1(G) = -[\hat{C}] \quad \text{and} \quad c_2(G) = 24 + \delta' + \deg_C Q.$$

Proof. (i) The idea is to construct a rank one, torsion free sheaf on \hat{C} whose degree is as large as possible. For $j \in \{1, \dots, \delta\}$ we get the pair of linear applications

$$q_{1j} : T_{S, \hat{x}_j} = (\nu^*T_S)_{x_{1j}} \longrightarrow Q_{x_{1j}} \cong \mathbb{C}, \quad q_{2j} : T_{S, \hat{x}_j} = (\nu^*T_S)_{x_{2j}} \longrightarrow Q_{x_{2j}} \cong \mathbb{C}.$$

For each index j we distinguish between two cases: either q_{1j}, q_{2j} are collinear or they are not.

Step (1) If q_{1j}, q_{2j} are collinear, that is $\text{Ker}(q_{1j}) = \text{Ker}(q_{2j})$, we find an isomorphism τ_j such that the diagram below commutes:

$$\begin{array}{ccc} T_{S, \hat{x}_j} & \xrightarrow{q_{1j}} & Q_{x_{1j}} \\ \parallel & & \downarrow \tau_j \\ T_{S, \hat{x}_j} & \xrightarrow{q_{2j}} & Q_{x_{2j}} \end{array}$$

Identifying $Q_{x_{1j}}$ and $Q_{x_{2j}}$ using τ_j we get an invertible sheaf on the nodal curve $C/\{x_{1j} = x_{2j}\}$ whose degree equals $\deg_C(Q)$.

We repeat this procedure until we eliminate all the pairs of points such that q_{1j}, q_{2j} are collinear. The result consists of a nodal curve C' , a naturally induced morphism $\nu' : C' \rightarrow \hat{C}$, and an invertible sheaf $Q' \rightarrow C'$ which is a destabilizing quotient for $(\nu')^*T_S \rightarrow C'$. We denote

$$\{x'_{11}, x'_{21}\}, \dots, \{x'_{1\delta'}, x'_{2\delta'}\} \subset C', \text{ where } 0 \leq \delta' \leq \delta,$$

the remaining points which are identified under the partial normalization $\nu' : C' \rightarrow \hat{C}$, and $q'_{11}, q'_{21}, \dots, q'_{1\delta'}, q'_{2\delta'}$ the corresponding projections.

Step (2) If q_{1j}, q_{2j} are not collinear, $\text{Ker}(q_{1j}) \oplus \text{Ker}(q_{2j}) = E_{\hat{x}_j}$, and is not possible perform the previous construction. Since $(\nu')^*E \rightarrow Q'$ is surjective, $\nu_*q : E \otimes \nu_*\mathcal{O}_C \rightarrow \hat{Q} := \nu_*Q'$ is the same. Observe that $\hat{Q} \rightarrow \hat{C}$ is a torsion free sheaf of rank one and degree $\deg_{\hat{C}} \hat{Q} = \deg_C Q + \delta'$.

We claim that the composition $\hat{q} : E \rightarrow \hat{Q}$ is still surjective. Indeed, consider a double point $\hat{x} \in \{\hat{x}_1, \dots, \hat{x}_{\delta'}\}$ of \hat{C} ; since the homomorphisms $q_1 : E_{\hat{x}} \rightarrow Q_{x_1}$ and $q_2 : E_{\hat{x}} \rightarrow Q_{x_2}$ are not collinear, we get the following local description:

$$k[[u, v]]^{\oplus 2} \otimes_{k[[u, v]]} \frac{k[[u, v]]}{\langle uv \rangle} \xrightarrow{\iota} k[[u, v]]^{\oplus 2} \otimes_{k[[u, v]]} \left(\frac{k[[u, v]]}{\langle v \rangle} \oplus \frac{k[[u, v]]}{\langle u \rangle} \right) \xrightarrow{\nu_*q} k[[u]] \oplus k[[v]], \quad \text{with}$$

$$\iota(f, g) = (f(u, 0) + f(0, v), g(u, 0) + g(0, v)) \quad \text{and}$$

$$\hat{q}(h_1(u) + k_1(v), h_2(u) + k_2(v)) = h_1(u) + k_2(v).$$

Our claim becomes now clear.

(ii) The fact that G is locally free is a consequence of the fact that \hat{C} is a complete intersection on S , and the formulae for the degree are obtained by straightforward computation (see for instance [7, Proposition 5.2.2]). \square

We are going to use this lemma to prove the following result:

Proposition 1.2. *Let S be a Picard general $K3$ surface, and denote by A the positive generator of $\text{Pic}(S)$. Consider a reduced, irreducible, nodal curve $\hat{C} \in |dA|$, for some $d \geq 1$, and let $\nu : C \rightarrow \hat{C}$ be its normalization. Let δ denote the number of double points of \hat{C} . If either*

- $d = 1, 2$ and $\delta \leq \frac{d^2 A^2}{4} - 25$ or
- $d \geq 3$ and $\delta \leq (d-1)A^2 - 23$

*holds, then $\nu^*T_S \rightarrow C$ is a stable, rank two bundle.*

Proof. Let us denote by A the positive generator of $\text{Pic}(S)$, and assume that $\hat{C} \in |dA|$. We denote as before $\{x_{11}, x_{21}\}, \dots, \{x_{1\delta}, x_{2\delta}\} \subset C$, with $\delta \leq p_a(\hat{C}) = (d^2 A^2 + 1)/2$, the pairs of points which are identified by ν .

Let us assume that $\nu^*T_S \rightarrow C$ is not stable. Then it has a destabilizing quotient $\nu^*T_S \rightarrow Q$ of rank one; in particular $\deg_C Q \leq 0$. We consider the torsion free sheaf constructed in the previous lemma, and the kernel $G := \text{Ker}(\hat{q} : T_S \rightarrow \hat{Q})$.

The assumption on δ implies that the discriminant

$$(1.1) \quad \Delta(G) := 4c_2(G) - c_1^2(G) = 4(24 + \delta' + \deg_C Q - \frac{d^2 A^2}{4}) \leq 4(24 + \delta' - \frac{d^2 A^2}{4}) < 0.$$

Using [7, theorem 7.3.4] we deduce the existence of a subsheaf $G' \subset G$ of rank one with torsion free quotient, having the following properties:

$$(1.2) \quad \xi_{G', G} := c_1(G') - \frac{c_1(G)}{2} > 0, \quad \text{and} \quad \xi_{G', G}^2 \geq \frac{-\Delta(G)}{4}.$$

The tangent bundle of a $K3$ surface is stable as it admits a Kähler-Einstein metric. This implies that $c_1(G') \cdot A < 0$; since S is a Picard general $K3$ surface, we find that $c_1(G') = -kA$

with $k \geq 1$. Further, the inequality $\xi_{G',G} > 0$ is equivalent to $d - 2k \geq 1$. In particular $d \geq 3$. For $d = 1$ or $d = 2$, we have already our contradiction which comes from the assumption that $\nu^*T_S \rightarrow C$ is not stable.

In higher degrees, we must go further a little bit, and use the second inequality in (1.2):

$$\frac{(d-2)^2 A^2}{4} \geq \frac{(d-2k)^2 A^2}{4} \geq \frac{d^2 A^2 - 4(\delta + 24)}{4}.$$

This implies that $\delta + 24 \geq (d-1)A^2$, which disagrees again with our hypothesis. \square

Remark 1.3. The result is unsatisfactory because it gives no insight into what happens for a larger number of nodes. For $d = 1, 2$, the bound $\frac{d^2 A^2}{4} - 25$ corresponds roughly to the half of the arithmetic genus of \hat{C} . For $d \geq 3$ the upper bound is very weak and does not seem to have any geometrical meaning.

The weakness in the previous proof is that in (1.1) we use only that $\deg_C Q \leq 0$, and that we have no control on δ' . If we restrict our attention to $T_S|_{\hat{C}}$, we obtain more precise informations.

Proposition 1.4. *Let S be a Picard general K3 surface with $\text{Pic}(S) = \mathbb{Z}A$, and consider a reduced, irreducible, nodal curve $\hat{C} \in |dA|$, for some $d \geq 1$. Let $n := \frac{A^2}{2} + 1$. If either*

- $d = 1$ and $n \geq 50$ or
- $d \geq 2$ and $(d-1)(n-1) \geq 13$,

is fulfilled, then the following statements hold:

- (i) $T_S|_{\hat{C}} \rightarrow \hat{C}$ has no non-trivial section;
- (ii) $T_S|_{\hat{C}} \rightarrow \hat{C}$ has no torsion free quotient of negative degree;
- (iii) $T_S|_{\hat{C}} \rightarrow \hat{C}$ is a simple, rank two bundle.

The inequalities in the lemma leave uncovered only finitely many cases, enumerated below:

n	2	3	4	5	6,7	8,...,13	14,...,49
d	1,...,13	1,...,7	1,...,5	1,...,4	1,2,3	1,2	1

Secondly, we observe that the exact sequences

$$\begin{aligned} 0 &\longrightarrow T_S \otimes A^{-d} \longrightarrow T_S \longrightarrow T_S \otimes \mathcal{O}_{\hat{C}} \longrightarrow 0, \quad \text{and} \\ 0 &\longrightarrow \text{End}(T_S) \otimes A^{-d} \longrightarrow \text{End}(T_S) \longrightarrow \text{End}(T_S) \otimes \mathcal{O}_{\hat{C}} \longrightarrow 0 \end{aligned}$$

imply that the statements are trivially satisfied for $d \gg 0$. The point of the proposition is that of making this result effective.

Proof. (i) We use the same notations as before. Let us assume to the contrary that there is a non-trivial section $\hat{\sigma} : \mathcal{O}_{\hat{C}} \rightarrow T_S \otimes \mathcal{O}_{\hat{C}}$. The induced section $\sigma : \mathcal{O}_C \rightarrow \nu^*T_S$ defines the line bundle $L = \mathcal{O}_C(\{\sigma = 0\}) \subset \nu^*T_S$, and the quotient $Q = \mathcal{O}_C(-\{\sigma = 0\})$ of negative degree.

This data determines the rank one, torsion free sheaf $\hat{Q} \rightarrow \hat{C}$ constructed in lemma 1.1. We use that σ is induced by a section in $T_S|_{\hat{C}}$ as follows: we divide the double points $\hat{x}_1, \dots, \hat{x}_\delta$ of \hat{C} into those where the section $\hat{\sigma}$ does not vanish and those where $\hat{\sigma}$ does vanish. Let again δ' denote the number of this latter set. For each such double point, the degree of Q goes down by 2. It follows that

$$\deg_{\hat{C}} \hat{Q} = \deg_C(Q) + \delta' \leq -2\delta' + \delta' = -\delta' \leq 0,$$

and therefore $\Delta(G) \leq 96 - d^2 A^2 < 0$. As before, we deduce the existence of a rank one subsheaf $G' \subset G$ such that (1.2) holds. For $d = 1, 2$ this is impossible, while for $d \geq 3$ we get

$$\frac{(d-2)^2 A^2}{4} \geq \frac{d^2 A^2 - 96}{4} \iff (d-1)A^2 \leq 24.$$

Again, this contradicts our assumption.

(ii) In this case the term $\delta' = 0$ in formula (1.1), so that $\Delta(G) < 0$. We conclude as before.

(iii) Let $\hat{\varphi} : T_S|_{\hat{C}} \rightarrow T_S|_{\hat{C}}$ be an automorphism; we wish to prove that is a multiple of the identity. Since $\det T_S \cong \mathcal{O}_S$, after multiplying $\hat{\varphi}$ with a constant, we may assume that $\det(\hat{\varphi}) = 1$. Let λ be an eigenvalue of $\hat{\varphi}$ at a smooth point $y \in \hat{C}$.

Consider the induced automorphism $\varphi : \nu^* T_S \rightarrow \nu^* T_S$, and define

$$E(\lambda) := \text{Ker}(\varphi - \lambda \text{id}) = \{w \in \nu^* T_S \mid \varphi(w) = \lambda w\} \subset \nu^* T_S.$$

It is a torsion free sheaf over C , and therefore locally free. We claim that it is the pull-back of a locally free sheaf on \hat{C} .

If $\text{rk } E(\lambda) = 2$ we are done. So assume that $\text{rk } E(\lambda) = 1$. Consider a double point $\hat{x} \in \hat{C}$, and $x_1, x_2 \in C$ with $\nu(x_1) = \nu(x_2) = \hat{x}$. We wish to prove that the diagram below commutes:

$$\begin{array}{ccc} E(\lambda)_{x_1} \subset (\nu^* T_S)_{x_1} & \xrightarrow{\varphi_{\hat{x}}} & (\nu^* T_S)_{x_1} \supset E(\lambda)_{x_1} \\ & \tau_{\hat{x}} \parallel & \parallel \tau_{\hat{x}} \\ E(\lambda)_{x_2} \subset (\nu^* T_S)_{x_2} & \xrightarrow{\varphi_{\hat{x}}} & (\nu^* T_S)_{x_2} \supset E(\lambda)_{x_1} \end{array}$$

Since $\varphi = \nu^* \hat{\varphi}$, certainly holds $\tau_{\hat{x}} \varphi_{\hat{x}} = \varphi_{\hat{x}} \tau_{\hat{x}}$. For $w \in E(\lambda)_{x_1}$ we have

$$\varphi_{\hat{x}}(\tau_{\hat{x}}(w)) = \tau_{\hat{x}}(\varphi_{\hat{x}}(w)) = \tau_{\hat{x}}(\lambda w) = \lambda \tau_{\hat{x}}(w).$$

So $E(\lambda)$ is preserved by the identifications $\tau_{\hat{x}}$, and descends to a locally free sheaf \hat{E} on \hat{C} of the same degree. Further, let $Q(\lambda) := \text{Im}(\varphi - \lambda \text{id}) \subset \nu^* T_S$. The $\tau_{\hat{x}}$'s preserve $Q(\lambda)$ too, so it descends to $\hat{Q} \rightarrow \hat{C}$ of the same degree. All together, we get the exact sequence

$$0 \longrightarrow \hat{E} \longrightarrow T_S|_{\hat{C}} \longrightarrow \hat{Q} \longrightarrow 0 \text{ over } \hat{C}, \text{ with } \deg_{\hat{C}} \hat{E} + \deg_{\hat{C}} \hat{Q} = 0.$$

Actually $\det(T_S|_{\hat{C}}) = \mathcal{O}_{\hat{C}}$ implies that $\hat{Q} = \hat{E}^\vee$. As $\hat{Q} \subset T_S|_{\hat{C}}$ is a sub-vector bundle, we deduce also the exact sequence $0 \rightarrow \hat{Q} \rightarrow T_S|_{\hat{C}} \rightarrow \hat{Q}^\vee = \hat{E} \rightarrow 0$ over \hat{C} . This means that both \hat{E} and \hat{Q} are quotients of $T_S|_{\hat{C}}$. Since the sum of their degrees is zero, at least one of \hat{E} or \hat{Q} has negative degree. We have already seen that this is impossible. \square

The result that is missing, which I am unable to address is:

Question 1.5. *Let S be a Picard general K3 surface, and consider a reduced, irreducible, nodal curve $\hat{C} \in |dA|$, for some $d \geq 1$. Let $\nu : C \rightarrow \hat{C}$ denote the normalization morphism, and assume that $C \not\cong \mathbb{P}^1$. Is it true that $\nu^* T_S \rightarrow C$ has no non-trivial section?*

Remark 1.6. Proposition 1.2 ties in with the work of Demailly, Peternell and Schneider in [2]. They call (see section 6 in *loc. cit.*) a vector bundle $\mathcal{V} \rightarrow X$ over a projective manifold *almost nef* if there is a countable family $A_i \hookrightarrow X$ of proper subvarieties of X such that $\mathcal{V}|_C \rightarrow C$ is nef for all curves $C \not\subset \bigcup_i A_i$. They have also introduced the somewhat stronger notion of *pseudo-effective* bundles; any pseudo-effective bundle is almost nef, but the converse is unclear.

For Calabi-Yau manifolds X , the authors proved in [2, corollary 6.12] that neither T_X nor Ω_X^1 is pseudo-effective. However, they leave open the question whether T_X or Ω_X^1 is almost nef or not (even for $K3$ surfaces); is speculated there that this should not be the case.

For a vector bundle $E \rightarrow C$ with $\det E \cong \mathcal{O}_C$ over a smooth curve, Miyaoka's criterion [9, theorem 3.1] says that E is nef if and only if E is semi-stable. Our proposition 1.2 can be interpreted as follows: assuming that the tangent bundle $T_S \rightarrow S$ of a $K3$ surface is not almost nef, the curves $\hat{C} \subset S$ on which it fails to be nef are quite singular. This might be viewed as a hint that T_S still enjoys some very weak positivity property.

2. RIGIDITY OF CURVES ON $K3$ SURFACES AND APPLICATIONS

The purpose of this section is to prove that nodal curves on $K3$ surfaces are rigid *within their conformal structure*, and to deduce the existence of low genus curves on projective $K3$'s. This represented actually the main motivation for this study.

The heuristic idea behind it is exactly the same as in the work of M. Green and Ph. Griffiths: for a polarized $K3$ surface (S, A) with $\text{Pic}(S) = \mathbb{Z}A$ holds

$$\dim |dA| = \frac{d^2 A^2}{2} + 1 =: p(d) = \text{genus of the general curve } C \in |dA|.$$

One expects that the (irreducible) nodal curves with δ double points form a δ -codimensional quasi-projective subvariety of $|dA|$, since each double point imposes (theoretically) one restriction. The difficulty consists in proving that things happen like this, and not else.

From now on, S will stay for a Picard general $K3$ surface, and A will denote the ample generator of $\text{Pic}(S)$. We fix integers $d \geq 1, \delta \geq 0$, and denote by

$$|dA|_\delta := \{C \in |dA| \mid C \text{ is reduced, irreducible, and its only singularities are } \delta \text{ nodes}\}.$$

In [11, Theorem 2.6] is proved that, if non-empty, $|dA|_\delta$ is a smooth, quasi-projective variety of dimension $p(d) - \delta$.

I make the following *two assumptions*:

$$(\star) \quad \begin{array}{ll} \text{it holds:} & \text{either } d = 1, 2 \quad \text{and} \quad \delta \leq \frac{d^2 A^2}{4} - 25 =: \delta_{\max} \\ & \text{or } d \geq 3 \quad \text{and} \quad \delta \leq (d-1)A^2 - 23 =: \delta_{\max}, \end{array}$$

and

$$(\star\star) \quad \text{every irreducible component of } \overline{|dA|}_\delta \setminus |dA|_\delta \text{ contains reduced curves.}$$

The second condition is trivially satisfied for $d = 1$, but I am unable to remove it in general. Under these assumptions, I can prove:

Proposition 2.1. *The linear system $|dA|$ contains a $(p(d) - \delta)$ -dimensional family corresponding to reduced and irreducible curves having precisely δ nodes.*

For proving the statement we are going to use two classical results: the first one is due to O. Zariski [12] (in the case of the projective plane), and gives an upper bound on the dimension of families of nodal curves on surfaces. The statement below is [1, Lemma 3.1].

Lemma *Let $U_{d,g} \subset |dA|$ be the subset consisting reduced and irreducible curves of geometric genus g . Then $\dim U_{d,g} \leq g$. If $g > 0$ and W is an irreducible component of $U_{d,g}$ of dimension g , then the general member of W is a nodal curve.*

The second result is due to S. Diaz, and gives an upper bound on the dimension of complete subvarieties of the (non-compactified) Deligne-Mumford space. He proves in [3] the following:

Theorem *The dimension of any complete subvariety of M_g is at most $g - 2$.*

I am going to use the language of stable maps (at a very elementary level); for the necessary definitions, notations and properties the reader may consult [5]. The advantage of this approach is that there are two natural morphisms

$$\begin{array}{ccc} \overline{M}_g(S, dA) & \xrightarrow{\Pi_g} & |dA| & [u : C \rightarrow S] & \longmapsto & \text{line corresponding to the} \\ & \downarrow \pi_g & & \downarrow & & \text{section defining } u_*C \\ \overline{M}_g & & & [C_{\text{stab}}] & & \end{array}$$

which permit to combine the properties of the linear system $|dA|$ with the geometry of \overline{M}_g ; this latter allows to control the genus of the normalization of the hyperplane sections. Observe there is no natural (overall defined) morphism $|dA| \rightarrow \overline{M}_{p(d)}$. We denote

$D_{g-\delta, 2\delta} := \text{Image}(\overline{M}_{g-\delta, \{x_{1j}, x_{2j}\}_{j=1, \dots, \delta}} \rightarrow \overline{M}_g)$, $\Delta_{g', g''} := \text{Image}(\overline{M}_{g', 1} \times \overline{M}_{g'', 1} \rightarrow \overline{M}_{g'+g''})$
and $\partial \overline{M}_g := D_{g-1, 2} \cup \bigcup_{g'+g''=g} \Delta_{g', g''}$ the ‘boundary divisor’.

Further, we let $\overline{M}_g^*(S, dA) \hookrightarrow \overline{M}_g(S, dA)$ be the (possibly empty) union of those irreducible components whose generic points are non-multiply covered (also called simple) stable maps, with domain of definition a smooth, irreducible curve of genus g .

For an irreducible component $W \hookrightarrow \overline{M}_g^*(S, dA)$, we denote $W^{**} \subset W$ the (possibly empty) Zariski open subset corresponding to morphisms whose image in S is an irreducible nodal curve with $p(d) - g$ nodes.

Lemma 2.2. *Let $r, d, d_1, \dots, d_r \geq 1$, with $d_1 + \dots + d_r = d$.*

Denote $\overline{M}_{g_1, \dots, g_r}^(S, (d_1, \dots, d_r)A) \hookrightarrow \overline{M}_g^*(S, dA)$ the union of those components whose generic points are simple stable maps, with domain of definition consisting of r curves of genera g_1, \dots, g_r , and degree d_j on the j^{th} component. Let $W \hookrightarrow \overline{M}_{g_1, \dots, g_r}^*(S, (d_1, \dots, d_r)A)$ be an irreducible component.*

Then $\dim W \leq g_1 + \dots + g_r$, with equality if the domain of definition of the generic point in W consists of r smooth curves, and its image is a nodal curve in S .

Proof. One applies inductively, r times, the previous lemma. □

Proposition 2.1 is a consequence of the following more precise statement, which describes the geometry of curves on general K3 surfaces.

Theorem 2.3. *Assume that (\star) and $(\star\star)$ holds. Then:*

(i) *The space $\overline{M}_{p(d)-\delta}^*(S, dA)$ is non-empty. It contains a component W of dimension $p(d) - \delta$, whose generic point represents a morphism $\nu : C \rightarrow S$ with the property that $\nu(C) \hookrightarrow S$ is an irreducible nodal curve.*

(ii) *The projection $\pi_{p(d)-\delta} : W \rightarrow \overline{M}_{p(d)-\delta}$ is generically finite.*

Proof. We prove by induction with respect to δ . For $\delta = 0$, the general point of $\overline{M}_{p(d)}^*(S, dA)$ corresponds to the general hyperplane section of $A^{\otimes d}$. In particular, $\overline{M}_{p(d)}^*(S, dA)$ is irreducible, of dimension $p(d)$. The infinitesimal deformations of the general morphism $\nu : C \rightarrow S$ relative to \overline{M}_g are contained in $H^0(C, T_S|_C)$. Since, by proposition 1.2, $T_S|_C \rightarrow C$ is a stable bundle of degree zero, it has no non-trivial sections. It follows that the projection $\pi_{p(d)}$ is generically finite.

Let us assume that both statements hold for $\delta < \delta_{\max}$. We are going to prove that they hold for $\delta + 1$. Write $g := p(d) - \delta$ for shorthand.

The first part of the induction hypothesis says that there is a component $W \hookrightarrow \overline{M}_g^*(S, dA)$ of dimension g ; its generic point corresponds to a stable map whose image is a nodal curve on S . The second hypothesis is that the projection π_g is quasi-finite, and therefore $\dim \pi_g(W) = g$. Diaz's theorem implies that the intersection $\pi_g(W) \cap \partial \overline{M}_g \neq \emptyset$. It follows that $W \setminus W^{**} \neq \emptyset$, and its components are divisors on W .

At this point, the assumption $(\star\star)$ guarantees the existence of a component $V_{\text{spl}} \hookrightarrow W \setminus W^{**}$ whose generic point represents a simple (that is non-multiply covered) stable map. The previous discussion implies that

$$\pi_g(V_{\text{spl}}) = \begin{cases} \pi_g(W) \cap D_{g-1,2} & \text{or} \\ \pi_g(W) \cap \Delta_{g',g''} & \text{with } g' + g'' = g. \end{cases}$$

Case (1) If $\pi_g(V_{\text{spl}}) = \pi_g(W) \cap D_{g-1,2}$, then $V_{\text{spl}} \hookrightarrow \text{Image}(\overline{M}_{g-1,2}(S, dA) \rightarrow \overline{M}_g^*(S, dA))$; let $\tilde{V}_{\text{spl}} \hookrightarrow \overline{M}_{g-1,2}(S, dA)$ be the component which projects onto V_{spl} , and let $[(C, u)] \in \tilde{V}_{\text{spl}}$ be the generic point: we wish to prove that C is irreducible and that $\hat{C} := u(C) \hookrightarrow S$ is a nodal curve. From the recursive construction of W , we deduce that we may view

$$\tilde{V}_{\text{spl}} \hookrightarrow \overline{M}_{p(d)-(\delta+1), \{x_{1j}, x_{2j}\}_{j=1, \dots, \delta+1}}(S, dA),$$

that is $[(C, u)]$ is obtained by identifying $\delta+1$ pairs of points. Let $\{C_j\}_{j=1, \dots, r}$ be the irreducible components of C , and let $\nu : \bigsqcup_{j=1}^r C_j \rightarrow C$ be the partial normalization (glueing) map. We consider the exact sequences

$$0 \rightarrow \mathcal{O}_C \rightarrow \nu_* \mathcal{O}_{\bigsqcup_{j=1}^r C_j} \rightarrow \mathcal{S}_C \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \mathcal{O}_{\hat{C}} \rightarrow (u \circ \nu)_* \mathcal{O}_{\bigsqcup_{j=1}^r C_j} \rightarrow \mathcal{S} \rightarrow 0.$$

\mathcal{S}_C is a skyscraper sheaf, supported at the double points where the irreducible components of C are glued together; at each such point the length of \mathcal{S}_C equals one.

Since u is a simple map, \mathcal{S} is a skyscraper sheaf too. It is supported at the various intersections of the $u(C_j)$'s; among these points appear the images of the $\delta + 1$ pairs of marked points on C , and also the images of the points where the C_j 's are glued together. It follows that $\text{length}(\mathcal{S}) \geq (\delta + 1) + \text{length}(\mathcal{S}_C)$.

Taking the cohomology of these sequences, we deduce:

$$g - 1 = h^1(\mathcal{O}_C) = \sum_{j=1}^r h^1(\mathcal{O}_{C_j}) + \text{length}(\mathcal{S}_C) - (r - 1) = \sum_{j=1}^r h^1(\mathcal{O}_{C_j}) + b_1(\Gamma_C), \quad \text{and} \\ \sum_{j=1}^r h^1(\mathcal{O}_{C_j}) = h^1(\mathcal{O}_{\hat{C}}) - \text{length}(\mathcal{S}) + (r - 1) = p(d) - \text{length}(\mathcal{S}) + (r - 1).$$

Here Γ_C denotes the graph of C , and $b_1(\Gamma_C)$ is its first Betti number. We obtain that $\text{length}(\mathcal{S}) = (\delta + 1) + \text{length}(\mathcal{S}_C)$. The previous remark implies that the equality holds precisely when the curves $u(C_j)$'s meet only at the (corresponding images) of the glueing points between the C_j 's, and at the images of the marked points.

We claim now that $b_1(\Gamma_C) = 0$, that is there are no loops in Γ_C . Otherwise there was a component of $\overline{M}_{h^1(\mathcal{O}_{C_1}), \dots, h^1(\mathcal{O}_{C_r})}(S, (d_j A)_j)$ of dimension $g - 1 > \sum_{j=1}^r h^1(\mathcal{O}_{C_j})$, in contradiction

to lemma 2.2. This argument also proves that none of $u|_{C_j}$ is constant, that is $d_j \geq 1$ for all j . If u is constant on some component, then we may omit it from the sum $\sum_{j=1}^r h^1(\mathcal{O}_{C_j})$.

So, Γ_C is a tree. The exact sequence $0 \rightarrow \mathcal{O}_{\hat{C}} \rightarrow u_*\mathcal{O}_C \rightarrow \mathcal{S}' \rightarrow 0$ implies that

$$p(d) - (\delta + 1) = h^1(\mathcal{O}_C) = h^1(\mathcal{O}_{\hat{C}}) - \text{length}(\mathcal{S}') \implies \text{length}(\mathcal{S}') = \delta + 1.$$

We have seen that the intersection pattern of the $u(C_j)$'s in S coincides with the intersection pattern of the C_j 's (which is encoded in Γ_C), and the intersections given by the identifications of the $\delta + 1$ pairs of double points. Since Γ_C is a tree, the r components C_1, \dots, C_r are glued together at $r - 1$ points; at these points there is no contribution to $\text{length}(\mathcal{S}')$. Assuming that $r \geq 2$, we deduce that

$$\begin{aligned} \delta + 1 &= \text{length}(\mathcal{S}') \geq \#\{\text{intersection points between the } u(C_j)\text{'s}\} - (r - 1) \\ &= \sum_{1 \leq i < j \leq r} d_i d_j A^2 - (r - 1) = \frac{d^2 - \left(\sum_{j=1}^r d_j^2\right)}{2} A^2 - (r - 1) \geq \frac{d^2 - [(d-r)^2 + (r-1)]}{2} A^2 - (r - 1) \\ &= \frac{r(2d-r-1)+1}{2} A^2 - (r - 1) \geq r \left[\frac{d-1}{2} A^2 - 1 \right] + \frac{1}{2} A^2 + 1 \geq \frac{2d-1}{2} A^2 - 1. \end{aligned}$$

This contradicts the assumption (\star) on δ . It follows that $r = 1$, that is C is irreducible, of genus $h^1(\mathcal{O}_C) = g - 1$. Since the map $[(C, u)]$ is the generic point of a $(g - 1)$ -dimensional family, C must be smooth and $u(C) \hookrightarrow S$ a nodal curve. The natural projection $\pi_{g-1} : \tilde{V}_{\text{spl}} \rightarrow \overline{M}_{g-1}$ is generically finite by proposition 1.2.

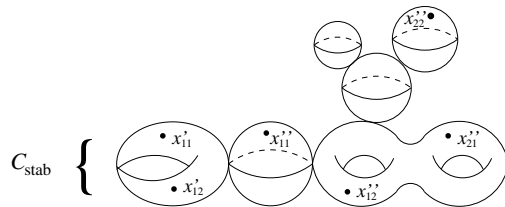
Case (2) If $\pi_g(V_{\text{spl}}) = \pi_g(W) \cap \Delta_{g',g''}$, then

$$V_{\text{spl}} \hookrightarrow \text{Image}(\overline{M}_{g',1}(S, d'A) \times \overline{M}_{g'',1}(S, d''A) \rightarrow \overline{M}_g^*(S, dA))$$

for some d', d'' , with $d' + d'' = d$. Let again $\tilde{V}_{\text{spl}} \hookrightarrow \overline{M}_{g',1}(S, d'A) \times \overline{M}_{g'',1}(S, d''A)$ be the component which projects onto V_{spl} , and $[(C, u)]$ be the generic point of \tilde{V}_{spl} . From the recursive construction of W , we deduce that the stabilized curve $[C_{\text{stab}}]$ lies in the fibred product

$$\begin{array}{ccc} \overline{M}_{g',\{\bullet\}} \times \overline{M}_{g'',\{\bullet\}} & \longleftarrow & \bigcup_{\delta'+\delta''=\delta} \overline{M}_{g'-\delta',\{\bullet\} \cup \{x'_{1j}, x'_{2j}\}_{j=1,\dots,\delta'}} \times \overline{M}_{g''-\delta'',\{\bullet\} \cup \{x''_{1j}, x''_{2j}\}_{j=1,\dots,\delta''}} \\ \downarrow & & \downarrow \\ \overline{M}_g & \longleftarrow & \overline{M}_{g-2\delta, \{x_{1j}, x_{2j}\}_{j=1,\dots,\delta}} \end{array}$$

It follows that $C_{\text{stab}} = C'_{\text{stab}} \cup C''_{\text{stab}}$, and these components carry δ' and δ'' pairs of marked points respectively. The curve C is obtained by attaching (bubble) trees of \mathbb{P}^1 's at smooth points of C'_{stab} and C''_{stab} , or at their glueing point. If it happens that such a tree is inserted at a marked point, this one 'jumps' on the tree.



In any case, $C = C' \cup C''$ and the restrictions $[(C', u' := u|_{C'})]$ and $[(C'', u'' := u|_{C''})]$ are (simple) stable maps to S ; let d' and d'' be their degrees respectively. The skyscraper sheaf \mathcal{S}' defined by the exact sequence

$$0 \rightarrow \mathcal{O}_{u'(C')} \rightarrow u'_* \mathcal{O}_{C'} \rightarrow \mathcal{S}' \rightarrow 0$$

is supported at least at the images of the δ' pairs of marked points. Observe that whenever different pairs of marked points are mapped to the same point in S , the length of \mathcal{S}' increases. It follows that $g' = h^1(\mathcal{O}_{C'}) = p(d') - \text{length}(\mathcal{S}') \leq p(d') - \delta'$. The same holds for g'' . All together we obtain:

$$p(d) - \delta = g = g' + g'' \leq (p(d') - \delta') + (p(d'') - \delta'') = p(d') + p(d'') - \delta \Rightarrow d' d'' A^2 \leq 1.$$

This is possible only if either d' or d'' vanishes. Assume that $d'' = 0$, that is $d' = d$.

We have found a component of $\overline{M}_{g'}(S, dA)$, with $g' \leq (p(d) - \delta) - 1$, whose dimension equals $p(d) - (\delta + 1)$ and whose generic point represents a simple map. Lemma 2.2 implies that necessarily $g' = p(d) - (\delta + 1)$.

We are now in the previous case, and we conclude that the domain of definition of $[(C', u|_{C'})]$ is smooth and irreducible, and that $u(C') \hookrightarrow S$ is a nodal curve. The same proposition 1.2 implies that the projection $\pi_{g-1} : \tilde{V}_{\text{spl}} \rightarrow \overline{M}_{g-1}$ is generically finite. \square

Remark 2.4. (i) For $d = 1$, the issue concerning the irreducibility of the domain of definition of the stable maps appearing in the proof is trivially satisfied. It would be very helpful to have a criterion to decide when is the limit of points in $|dA|_\delta$ a reduced and irreducible curve, or whether at least this situation appears generically.

(ii) A very simple example of a *theoretically possible* scenario, which we exclude assuming $(\star\star)$, is the following: suppose that for some genus g and (even) degree d , a component $W \hookrightarrow \overline{M}_g^*(S, dA)$ enjoys the properties in theorem 2.3; it is *a priori* possible that every boundary component of $W \setminus W^{**}$ (which is a divisor in W) consists 2-sheeted coverings.

The Riemann-Hurwitz formula implies that for a 2-sheeted covering $C \rightarrow D$ of smooth curves, ramified at ϱ points, holds: $g = g(C) = 2g(D) - 1 + \frac{\varrho}{2}$. On the other hand, the choice of the ramification points on D gives a ϱ -dimensional component in $\overline{M}_g(D, 2[D])$. So, its dimension grows faster than g .

Back to our situation, if $C \xrightarrow{u} S$ is a multiply-covered curve, it factorizes through $C \rightarrow D \xrightarrow{\tilde{u}} S$. The phenomenon described above prevents to get informations about the genus of D (the domain of the underlying simple stable map) only in terms of the dimension of W , to which the multiply-covered map u belongs.

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