QUOTIENTS OF AFFINE SPACES FOR ACTIONS OF REDUCTIVE GROUPS

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Abstract. The article studies actions of reductive groups on affine spaces. We prove the existence of a fan structure on the space of characters of the group, which parameterizes the possible invariant quotients. Secondly, we compute the Chow and the cohomology rings of the quotients. Further, we prove the vanishing of the higher cohomology groups of nef line bundles on the quotients. Finally we show that for 'large' G-modules V, the quotient morphism $\mathbb{V}^{\rm ss} \to \mathbb{V}/\!\!/G$ is a semi-stable principal G-bundle.

In this article we study the geometric properties of the varieties which are obtained as invariant quotients of affine spaces by the action of reductive groups. There are a multitude of examples of such varieties: the most at hand are the toric varieties, obtained for representations of tori, and the quiver varieties, which include as very particular cases the flag varieties of the general linear group.

The possible linearizations of the action of a linear group on an affine space are given by the characters of the group. A first issue that we address is the existence of a fan structure on the space of characters, whose cones parameterize the invariant quotients which appear.

Theorem *Let X be a normal, affine G-variety. The* GIT*-equivalence classes in $\mathcal{X}^*(G)_{\mathbb{R}}$, corresponding to the linearization of the G-action on X in $\mathscr{O}_X$, are the relative interiors of the cones of a rational, polyhedral fan $\Delta^G(X)$, which we call the* GIT*-fan of X.*

This result can be viewed as a non-abelian generalization of the Gelfand-Kapranov-Zelevinskij decomposition for linear representations of tori. The case when V is the representation space of a quiver has been treated in [16]. The analogous issue for projective varieties has been addressed in [7, 24].

In the rest of the paper we have investigated the geometrical properties of the quotients of an affine space $\mathbb{V}$ on which G acts linearly. The first issue in this direction is that of computing the Chow ring of the invariant quotients.

Theorem *Let $\chi \in \mathcal{X}^*(G)$ be a character with the property that the corresponding semi-stable locus coincides with the properly stable one. Then*

$$A_*\big(\mathbb{V}/\!\!/_{\chi}G\big)_{\mathbb{Q}} \cong \big(A_*^G\big)_{\mathbb{Q}} \Big/ \wp\big\langle [\mathbb{E}]_T\,;\, \mathbb{E} \textit{ is a } (T,\chi)\textit{-unstable component of } \mathbb{V}\big\rangle_{\mathbb{Q}},$$

where $\wp$ denotes the projection from the T- to the G-equivariant Chow ring of a point. If moreover the ground field is that of the complex numbers and $\mathbb{V}/\!\!/_{\chi}G$ is projective, then its cohomology ring is isomorphic to the Chow ring.

This formula has been obtained in [10] under the additional assumptions that the group G acting on the affine space contains its homotheties, and the stability concept corresponds to a character with large weight on the subgroup of homotheties. We also answer in positive a

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question raised in *loc. cit.*, asking whether the Chow ring of the quotients is generated by the Chern classes of 'natural' vector bundles, induced by representations of the group G.

In section 6 we use the techniques from [25] to show that projective varieties of the form $\mathbb{V}/\!\!/_{\chi}G$ fit together in families over separated and reduced schemes over $\operatorname{Spec}\mathbb{Z}$, and moreover that the absolute case considered in the previous sections corresponds to base change. In particular, we obtain flat families of quotients over $\operatorname{Spec}\mathbb{Z}$. This eventually allows us to extend in positive characteristic certain cohomological properties holding in characteristic zero. In theorem 6.2 we compute the Chow ring of families of such varieties, parameterized by reduced schemes defined over algebraically closed fields, and also, in the case of complex numbers, the cohomology ring of the total space of the fibration.

We conclude the article by computing the Picard group and the ample cone of the geometric quotients that we obtain. Using the Hochster-Roberts theorem, we further prove the vanishing of higher cohomology groups of nef line bundles.

Theorem *Let $\chi \in \mathcal{X}^*(G)$ be a character such that the codimension of the unstable locus $\operatorname{codim}_{\mathbb{V}}\mathbb{V}^{\rm us}(G,\chi) \geqslant 2$, and $\mathbb{V}^{\rm ss}(G,\chi) = \mathbb{V}^{\rm s}_{(0)}(G,\chi)$. There is a prime $p(\chi)$ depending only on the* GIT*-chamber of χ, having the property that over ground fields of characteristic zero or larger than $p(\chi)$:*
(i) *$\mathbb{V}/\!\!/_{\chi}G$ is arithmetically Cohen-Macaulay with respect to any ample, invertible sheaf on it;*
(ii) *For any nef, invertible sheaf $\mathscr{L} \to \mathbb{V}/\!\!/_{\chi}G$ holds: $H^i\big(\mathbb{V}/\!\!/_{\chi}G, \mathscr{L}\big) = 0,\ \forall\, i > 0$.*

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1. Linear actions of reductive groups: Basic facts

In this section we introduce part of the notations used in this paper, and also recall from [15, 20] some facts concerning linear actions of reductive groups.

Let K be an algebraically closed field of arbitrary characteristic, G a reductive group over K, and $G_m := \operatorname{Spec} K[t,t^{-1}]$ the multiplicative group of K. We denote $Z(G)^\circ$ the connected component of the centre of G; we fix a maximal torus T of G and denote by $W := N_G(T)/T$ the corresponding Weyl group. Further, we consider a finite dimensional G-module V, such that the representation $\rho : G \to \mathrm{Gl}(V)$ has finite kernel.

Let $\mathbb{V} := \operatorname{Spec}\big(\operatorname{Sym}^\bullet V^\vee\big)$ be the affine space corresponding to V, and consider

$$\Sigma : G \times \mathbb{V} \longrightarrow \mathbb{V}$$

the natural G-action on it (see [22], section 1.1). Since $\operatorname{Pic}(\mathbb{V}) = \{\mathscr{O}_{\mathbb{V}}\}$, the possible linearizations of Σ are parameterized by the group of characters $\mathcal{X}^*(G)$ of G: $\chi \in \mathcal{X}^*(G)$ defines the linearization

$$\Sigma_\chi : G \times \big(\mathbb{V} \times \mathbb{A}^1_K\big) \longrightarrow \mathbb{V} \times \mathbb{A}^1_K, \quad \Sigma_\chi\big(g,(x,z)\big) := \big(\Sigma(g,x), \chi(g)z\big). \tag{1.1}$$

A function $f \in K[\mathbb{V}]$ is Σ_χ-invariant if $f\big(\Sigma(g,x)\big) = \chi(g)f(x)$ for all $(g,x) \in G \times \mathbb{V}$. For $n \geqslant 0$, we denote by $K[\mathbb{V}]^G_{\chi^n}$ the vector space of Σ_{χ^n}-invariant functions on $\mathbb{V}$, and define the algebra

$$K[\mathbb{V}]^{G,\chi} := \bigoplus_{n\geqslant 0} K[\mathbb{V}]^G_{\chi^n}.$$

A one-parameter subgroup (1-PS for short) of G $\lambda \in \mathcal{X}_*(G)$ decomposes V into the direct sum of its weight spaces $V = \oplus_j V_j$, where $\lambda(t)|_{V_j} = t^j \mathrm{Id}_{V_j}$. This decomposition breaks $\mathbb{V}$ into the direct product

$$\mathbb{V} = \mathrm{Spec}\big(\mathrm{Sym}^\bullet V\big) = \mathrm{Spec}\big(\otimes_j \mathrm{Sym}^\bullet V_j\big) = \times_j \mathbb{V}_j,$$

and we remark that λ acts on $\mathbb{V}_j$ by multiplication through t^j. Let $(x_j)_j$ be the components of a point $x \in \mathbb{V}$ with respect to this decomposition, and we define

$$m(x,\lambda) := \begin{cases} \min\{j \mid x_j \neq 0\} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Lemma 1.1. (i) *For $\chi \in \mathcal{X}^*(G)$, the corresponding χ-unstable locus is*

$$\mathbb{V}^{\mathrm{us}}(G,\chi) = \bigcup_{\substack{\lambda \in \mathcal{X}_*(T),\\ \langle \chi,\lambda\rangle < 0}} G\cdot \mathbb{E}(\lambda) = G\cdot \mathbb{V}^{\mathrm{us}}(T,\chi),$$

where $\mathbb{E}(\lambda) := \{x \in \mathbb{V} \mid m(x,\lambda) \geqslant 0\}$.
(ii) *For $\lambda \in \mathcal{X}_*(G)$, $\mathbb{E}(\lambda)$ is a linear subspace of $\mathbb{V}$ which is stable under the parabolic subgroup*

$$P(\lambda) := \big\{g \in G \mid \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists in } G\big\}$$

of G. As a consequence, $G\cdot\mathbb{E}(\lambda)$ is closed in $\mathbb{V}$.
(iii) *There is a finite W-invariant set $\mathcal{F}(\chi) \subset \{\langle\chi,\cdot\rangle < 0\}$, such that*

$$\bigcup_{\substack{\lambda \in \mathcal{X}_*(T),\\ \langle \chi,\lambda\rangle < 0}} G\cdot \mathbb{E}(\lambda) = \bigcup_{\lambda\in\mathcal{F}(\chi)} G\cdot\mathbb{E}(\lambda).$$

Proof. Is a particular case of [15], applied to the action induced by $\rho' : G \to \mathrm{Gl}(K\oplus V)$, $\rho'(g) := \mathrm{diag}\big(\chi^{-1}(g), \rho(g)\big)$. □

The Hilbert-Mumford criterion corresponding to our linear problem is the following:

Proposition 1.2. *Fix a character $\chi \in \mathcal{X}^*(G)$. Then:*
(i) $x \in \mathbb{V}^{\mathrm{ss}}(G,\chi) \iff \big[\forall\,\lambda \in \{\langle\chi,\cdot\rangle < 0\} \Rightarrow m(x,\lambda) < 0\big]$.
(ii) $x \in \mathbb{V}^{\mathrm{s}}_{(0)}(G,\chi) \iff \big[\forall\,\lambda \in \{\langle\chi,\cdot\rangle \leqslant 0\} \Rightarrow m(x,\lambda) < 0\big]$.

Proof. See [20]. □

2. Linear actions of reductive groups: reduction to the projective case

In this section we will view the affine space $\mathbb{V}$ as a G-invariant, Zariski open subset of the projective space $\mathbb{P}(K\oplus V) := \mathrm{Proj}\big(\mathrm{Sym}^\bullet(K\oplus V^\vee)\big)$, for an appropriate G-action on this latter one. For any character $\chi \in \mathcal{X}^*(G)$, we prove that the G-action on $\mathbb{P}(K\oplus V)$ can be linearized in such a way that the intersection of the corresponding semi-stable locus with $\mathbb{V}$ coincides with the χ-semi-stable locus of $\mathbb{V}$. Our goal is to reduce the issue of chamber structure on $\mathcal{X}^*(G)$ in the affine case to known results in the projective case.

Let us recall that for any character $\chi \in \mathcal{X}^*(G)$, the invariant quotient $\mathbb{V}/\!\!/_\chi G$ corresponding to the linearization Σ_χ can be described as $\mathrm{Proj}\big(K[\mathbb{V}]^{G,\chi}\big)$, and this one is projective over

$\operatorname{Spec}\big(K[\mathbb{V}]^G\big)$. We denote $\pi:\mathbb{V}=\operatorname{Spec}K[\mathbb{V}]\to\operatorname{Spec}K[\mathbb{V}]^G=:\mathbb{V}/G$ the natural projection, and define $\hat{0}:=\pi(0)\in\mathbb{V}/G$. Since $K[\mathbb{V}]^{G,\chi}$ is a subalgebra of $K[\mathbb{V}]$, it inherits an additional graduation given by the degree in $K[\mathbb{V}]$ (it becomes a bi-graded algebra); we denote by $K[\mathbb{V}]^G_{\chi^n,(p)}\subset K[\mathbb{V}]^G_{\chi^n}$ the submodule of homogeneous elements of degree p.

Lemma 2.1. *For a character $\chi\in\mathcal{X}^*(G)$, the following statements hold:*
(i) *There is an integer $c>0$ such that $K[\mathbb{V}]^G_{\chi^{nc}}=\big(K[\mathbb{V}]^G_{\chi^c}\big)^n$ for all $n\geqslant 1$.*
Assume that $\chi\in\mathcal{X}^*(G)$ is such that the condition (i) *holds for $c=1$.*
($\mathrm{ii_a}$) *In the case $K[\mathbb{V}]^G\neq K$, there are integers $D,D_\chi\geqslant 1$ such that the following two conditions are fulfilled:*

$$\begin{cases}\forall n\geqslant 1\,\forall x\in\mathbb{V}\setminus\pi^{-1}(\hat{0})\;\exists f\in K[\mathbb{V}]^G_{(nD)}\text{ with } f(x)\neq 0;\\ \forall n>D_\chi\,\forall x\in\mathbb{V}^{\mathrm{ss}}(G,\chi)\setminus\pi^{-1}(\hat{0})\;\exists f\in K[\mathbb{V}]^G_{\chi^D,(nD)}\text{ with } f(x)\neq 0.\end{cases}$$

Conversely: consider $x\in\mathbb{V}$ with the property that there are $N>0$ and $f\in K[\mathbb{V}]^G_{\chi^N,(Nd)}$, with $d>D_\chi$, such that $f(x)\neq 0$. Then $x\in\mathbb{V}^{\mathrm{ss}}(G,\chi)\setminus\pi^{-1}(\hat{0})$.
($\mathrm{ii_b}$) *In the case $K[\mathbb{V}]^G=K$,*

$$\max\big\{\deg f\mid f\in K[\mathbb{V}]^G_{\chi^n}\big\}\leqslant n\cdot\max\big\{\deg f\mid f\in K[\mathbb{V}]^G_{\chi}\big\}=:nD_\chi,\ \textit{for all } n\geqslant 1.$$

Proof. The first statement is a particular case of [12], lemme 2.1.6. The statement ($\mathrm{ii_b}$) is obvious, since in this case $K[\mathbb{V}]^G_\chi$, which is a finite dimensional K-module, generates $K[\mathbb{V}]^{G,\chi}$ as a K-algebra.

We prove now ($\mathrm{ii_a}$): we consider finite sets of homogeneous polynomials $\mathcal{S}_0\subset K[\mathbb{V}]^G\setminus K$ and $\mathcal{S}_1\subset K[\mathbb{V}]^G_\chi$ which generate respectively $K[\mathbb{V}]^G$ as a K-algebra, and $K[\mathbb{V}]^{G,\chi}$ as a $K[\mathbb{V}]^G$-algebra. We define D to be the smallest common multiple of $\deg f$, for $f\in\mathcal{S}_0$, and $D_\chi:=\max\{\deg f_1\mid f_1\in\mathcal{S}_1\}$. For $x\in\mathbb{V}\setminus\pi^{-1}(\hat{0})$ there is $f\in\mathcal{S}_0$ such that $f(x)\neq 0$. Raising it to a suitable power, we obtain the desired polynomial.

Consider an integer $n\geqslant 1$. For $x\in\mathbb{V}^{\mathrm{ss}}(G,\chi)\setminus\pi^{-1}(\hat{0})$, we find $f_1\in\mathcal{S}_1$ such that $f_1(x)\neq 0$. The previous step shows that there is a homogeneous polynomial $f_0\in K[\mathbb{V}]^G$ of degree $\deg f_0=(n+D_\chi-\deg f_0)D$ with $f_0(x)\neq 0$. We obtain that

$$f_0f_1^D\in K[\mathbb{V}]^G_{\chi^D},\ \deg(f_0f_1^D)=\big(n+D_\chi\big)D,\ \text{and } f_0f_1^D(x)\neq 0.$$

Conversely, assume that $x\in\mathbb{V}$ has the stated property. Then, clearly, $x\in\mathbb{V}^{\mathrm{ss}}(G,\chi)$. We claim that the elements of $K[\mathbb{V}]^G\setminus K$ do not vanish at x. Assume the contrary, and consider a homogeneous decomposition $f=f'+f''$ with $f'\in K[s;s\in\mathcal{S}_1]$ and $f''\in\mathcal{I}[s;s\in\mathcal{S}_1]$, for $\mathcal{I}:=\langle s_0;s_0\in\mathcal{S}_0\rangle\subset K[\mathbb{V}]^G$. Since $f(x)\neq 0$, $f'(x)\neq 0$ too. Thus we have found a non-zero $f'\in K[\mathbb{V}]^G_{\chi^N,(Nd)}$ which can be expressed as a polynomial of degree N (because of the χ^N-homogeneity) in the elements of $\mathcal{S}_1$. But, in this case

$$Nd=\deg f'\leqslant N\cdot\max\{\deg f_1\mid f_1\in\mathcal{S}_1\}=ND_\chi,$$

which contradicts the choice of $d>D_\chi$. □

We will view $\mathbb{P}(K\oplus V)$ as the quotient $\big((\mathbb{A}^1_K\times\mathbb{V})\setminus\{0\}\big)/G_m$. Consider the representation $G\to\mathrm{Gl}(K\oplus V)$, $g\longmapsto\operatorname{diag}\big(1,\rho(g)\big)$, and observe that for the induced action

$$\bar{\Sigma}:G\times\mathbb{P}(K\oplus V)\longrightarrow\mathbb{P}(K\oplus V),\quad \bar{\Sigma}(g,[a,x]):=[a,\Sigma(g,x)],\tag{2.1}$$

the inclusion $\mathbb{V} \subset \mathbb{P}(K \oplus V)$ is G-invariant. We wish to linearize the G-action in $\mathscr{O}_{\mathbb{P}(K\oplus V)}(d)$, for some $d > 0$, such that the corresponding (semi-)stable points in $\mathbb{V}$ are precisely the Σ_χ-(semi-)stable points of $\mathbb{V}$. The line bundle defined by $\mathscr{O}_{\mathbb{P}(K\oplus V)}(d)$ can be described as

$$\mathcal{O}_{\mathbb{P}(K\oplus V)}(d) := \underline{\mathrm{Spec}}\big(\mathrm{Sym}^\bullet \mathscr{O}_{\mathbb{P}(K\oplus V)}(-d)\big) \cong \big((\mathbb{A}^1_K \times \mathbb{V}) \setminus \{0\}\big) \times \mathbb{A}^1_K \big/ G_m,$$

where G_m acts on $(\mathbb{A}^1_K \times \mathbb{V}) \times \mathbb{A}^1_K$ by $t \times (y, z) := (ty, t^d z)$ $\forall t \in G_m$. For $\chi \in \mathcal{X}^*(G)$, we linearize the G-action on $\mathbb{P}(K \oplus V)$ in $\mathscr{O}_{\mathbb{P}(K\oplus V)}(d)$ by

$$\Sigma_{d,\chi} : G \times \mathcal{O}_{\mathbb{P}(K\oplus V)}(d) \longrightarrow \mathcal{O}_{\mathbb{P}(K\oplus V)}(d), \quad \big(g, \big[(a, x), z\big]\big) \longmapsto \big[(a, \Sigma(g, x)), \chi(g)z\big]. \tag{2.2}$$

In particular, we deduce the natural isomorphism

$$\mathrm{Pic}^G\big(\mathbb{P}(K \oplus V)\big) \cong \mathrm{Pic}\big(\mathbb{P}(K \oplus V)\big) \oplus \mathcal{X}^*(G) \cong \mathbb{Z} \oplus \mathcal{X}^*(G). \tag{2.3}$$

Lemma 2.2. (i) *The action $\Sigma_{d,\chi}$ on $\mathcal{O}_{\mathbb{P}(K\oplus V)}(d)|_\mathbb{V}$ induces the action Σ_χ on $\mathcal{O}_\mathbb{V}$.*
(ii) *The restriction homomorphism $\mathrm{Pic}^G\big(\mathbb{P}(K \oplus V)\big) \longrightarrow \mathrm{Pic}^G(\mathbb{V})$ is surjective, and corresponds to the projection $\mathbb{Z} \oplus \mathcal{X}^*(G) \to \mathcal{X}^*(G)$.*

Proof. We consider the section s in $\mathcal{O}_{\mathbb{P}(K\oplus V)}(d)$ given by $s[a, x] := [(a, x), a^d]$; it defines an isomorphism $\mathcal{O}_\mathbb{V} \xrightarrow{\cong} \mathcal{O}_{\mathbb{P}(K\oplus V)}(d)|_\mathbb{V}$, and we observe that $\Sigma_{d,\chi}\big(g, s[1, x]\big) = \chi(g) \cdot s[1, \Sigma(g, x)]$ for $x \in \mathbb{V}$ and $g \in G$. □

We are going to clarify the behaviour of the (semi-)stable loci under restriction.

Proposition 2.3. *Let $\chi \in \mathcal{X}^*(G)$ be a character. There are positive integers c, D_χ depending on χ such that for any $d > D_\chi$ we have*

$$\mathbb{P}(K \oplus V)^{\mathrm{ss}}\big(\Sigma_{d,\chi^c}\big) = \mathbb{V}^{\mathrm{ss}}(G, \chi) \cup \big[\mathbb{V}^{\mathrm{ss}}(G, \chi) \setminus \pi^{-1}(\hat{0})\big]\big/G_m.$$

For writing this union, we view $\mathbb{P}(K \oplus V) = \mathbb{V} \cup \mathbb{P}(V)$. Moreover,

$$\mathbb{V} \cap \mathbb{P}(K \oplus V)^{\mathrm{s}}_{(0)}\big(\Sigma_{d,\chi^c}\big) = \mathbb{V}^{\mathrm{s}}_{(0)}(G, \chi).$$

In particular, if $K[\mathbb{V}]^G = K$, it follows that $\mathbb{P}(K \oplus V)^{\mathrm{ss}}\big(\Sigma_{d,\chi^c}\big) = \mathbb{V}^{\mathrm{ss}}(G, \chi)$.

Proof. We choose $c > 0$ such that the condition (i) in lemma 2.1 is fulfilled. Since the right-hand-side of the equality is invariant under the change $\chi \rightsquigarrow \chi^c$, we may assume that $K[\mathbb{V}]^{G,\chi}$ is generated by $K[\mathbb{V}]^G_\chi$ over $K[\mathbb{V}]^G$. We let $D, D_\chi > 0$ be as in ($\mathrm{ii_a}$), resp. ($\mathrm{ii_b}$), of *loc. cit.*

For $x \in \mathbb{V}^{\mathrm{ss}}(G, \chi)$, there is a homogeneous $f \in K[\mathbb{V}]^G_\chi$, with $\deg f \leqslant D_\chi$, such that $f(x) \neq 0$. The section

$$s \in \Gamma\big(\mathbb{P}(K \oplus V), \mathcal{O}_{\mathbb{P}(K\oplus V)}(d)\big) \text{ defined by } s[a, x] := \big[(a, x), a^{d-D_\chi} f(x)\big] \tag{2.4}$$

is clearly G-invariant and does not vanish at $[1, x]$.

Assume that $K[\mathbb{V}]^G \neq K$, and take $x \in \mathbb{V}^{\mathrm{ss}}(G, \chi) \setminus \pi^{-1}(\hat{0})$; we have proved that there is $f \in K[\mathbb{V}]^G_{\chi^D,(dD)}$, with $f(x) \neq 0$. The section $s \in \Gamma\big(\mathbb{P}(K \oplus V), \mathcal{O}_{\mathbb{P}(K\oplus V)}(dD)\big)$ defined by $s[a, x] := \big[(a, x), f(x)\big]$ is again G-invariant and does not vanish at $[0, x]$.

For the converse, we notice that lemma 2.2 implies that $\mathbb{V} \cap \mathbb{P}(K \oplus V)^{\mathrm{ss}}(\Sigma_{d,\chi}) \subset \mathbb{V}^{\mathrm{ss}}(G, \chi)$. If $K[\mathbb{V}]^G \neq K$, the converse to the condition ($\mathrm{ii_a}$) in lemma 2.1 says precisely that

$$\mathbb{P}(V) \cap \mathbb{P}(K \oplus V)^{\mathrm{ss}}\big(\Sigma_{d,\chi}\big) \subset \big[\mathbb{V}^{\mathrm{ss}}(G, \chi) \setminus \pi^{-1}(\hat{0})\big]/G_m$$

When $K[\mathbb{V}]^G = K$, we prove that $\mathbb{P}(V) \hookrightarrow \mathbb{P}(K \oplus V)$ consists of $\Sigma_{d,\chi}$-unstable points. Assume that $[0, x_0]$ is semi-stable; then there is $N > 0$ and $F \in \mathrm{Sym}^{Nd}\big((K \oplus V)^\vee\big)$ such that

$F(a,\Sigma(g,x)) = \chi^N(g)F(a,x)$ and $F(0,x_0) \neq 0$. Define $f(x) := F(0,x)$. Since $F(0,x_0) \neq 0$ it follows $\deg f = \deg F = Nd$. On the other hand, $f(\Sigma(g,x)) = \chi^N(g)f(x)$, that is $f \in K[\mathbb{V}]^G_{\chi^N}$. Applying (ii$_\mathrm{b}$) of lemma 2.1, we find that $\deg f \leqslant ND_\chi < Nd$, a contradiction.

For the second equality, notice that the left-hand- is obviously included in the right-hand-side. Conversely, for $x \in \mathbb{V}^{\mathrm{s}}_{(0)}(G,\chi)$ the section defined by (2.4) vanishes on the hyperplane $\mathbb{P}(V) \hookrightarrow \mathbb{P}(K \oplus V)$, and therefore the orbit $G[1,x]$ is closed in $\mathbb{P}(K \oplus V)^{\mathrm{ss}}(\Sigma_{d,\chi})$. □

Remark 2.4. By taking finite covers of G we can linearize the G-action in $\mathcal{O}_{\mathbb{P}(K\oplus V)}(1)$ without affecting the semi-stable locus. Indeed, for $d > 0$, we consider $\zeta_d : Z(G)^\circ \to Z(G)^\circ$, $z \mapsto z^d$, and define

$$m_d : Z(G)^\circ \times [G,G] \xrightarrow{(\zeta_d, \mathrm{id}_{[G,G]})} Z(G)^\circ \times [G,G] \xrightarrow{m_1} G,$$

where m_1 is the multiplication. For all $\chi \in \mathcal{X}^*(G)$ holds $\mathbb{V}^{\mathrm{ss}}(G,\chi) = \mathbb{V}^{\mathrm{ss}}(Z(G)^\circ \times [G,G], m_d^*\chi)$. Moreover, the representation

$$Z(G)^\circ \times [G,G] \to \mathrm{Gl}(K \oplus V), \quad (z,g') \longmapsto \mathrm{diag}\big(\chi^{-1}(z), \chi^{-1}(z)\cdot(\rho \circ m_d)(z,g')\big)$$

induces a linearization of the $Z(G)^\circ \times [G,G]$-action $\bar{\Sigma} \circ m_d$ in $\mathscr{O}_{\mathbb{P}(K\oplus V)}(1)$. For shorthand, we will write $g \mapsto \big(\chi(g)^{-1/d}, \chi(g)^{-1/d}\rho(g)\big)$ for this representation.

Following [7], subsection 1.1.3, we introduce numerical functions needed to address the issue of the chamber structure on $\mathcal{X}^*(G)$ with respect to the GIT-equivalence relation. We fix a norm on $\mathcal{X}_*(T)_\mathbb{R}$ invariant under the action of the Weyl group W; it naturally induces a norm $|\cdot|$ on $\mathcal{X}_*(G)$. Consider $x \in \mathbb{V}$ and $\lambda \in \mathcal{X}_*(G)$: an immediate computation shows that

$$\bar{\Sigma}\big(\lambda(t),[1,x]\big) = \big[1,(t^j x_j)_j\big] \xrightarrow{t\to 0} \begin{cases} [0, x_{m(v,\lambda)}] & \text{if } m(x,\lambda) < 0; \\ [1, x_0] & \text{if } m(x,\lambda) = 0; \\ [1,0] & \text{if } m(x,\lambda) > 0. \end{cases}$$

The weight of the λ-action at the specialization, corresponding to $\Sigma_{d,\chi}$, is

$$\mu_{d,\chi}(x,\lambda) = \begin{cases} \langle \chi, \lambda \rangle - d \cdot m(x,\lambda) & \text{if } m(x,\lambda) < 0; \\ \langle \chi,\lambda \rangle & \text{if } m(x,\lambda) \geqslant 0. \end{cases}$$

We extend $\mu_{\cdot,\cdot}(x,\lambda)$ to $\mathbb{R} \oplus \mathcal{X}^*(G)_\mathbb{R}$, and define the numerical functions

$$\begin{aligned} &M : \mathcal{X}^*(G)_\mathbb{R} \times \mathbb{V} \to \mathbb{R}, \quad M(\chi,x) := \inf\left\{ \frac{\langle \chi,\lambda\rangle}{|\lambda|} \;\middle|\; \begin{array}{l} \lambda \in \mathcal{X}_*(G) \\ m(x,\lambda) \geqslant 0 \end{array} \right\}, \\ &\bar{M} : \big(\mathbb{R} \oplus \mathcal{X}^*(G)_\mathbb{R}\big) \times \mathbb{V} \to \mathbb{R}, \quad \bar{M}\big((d,\chi),x\big) := \inf_{\lambda \in \mathcal{X}_*(G)} \frac{\mu_{d,\chi}(x,\lambda)}{|\lambda|}. \end{aligned} \tag{2.5}$$

3. The chamber structure on the cone of effective characters

There are two questions which naturally arise in the study of the quotients $\mathbb{V}/\!\!/_\chi G$, as $\chi \in \mathcal{X}^*(G)$ varies:

(i) Does it exist a chamber structure on $\mathcal{X}^*(G)$ corresponding to the possible semi-stable loci $\mathbb{V}^{\mathrm{ss}}(G,\chi)$?

(ii) For which characters χ is the corresponding semi-stable locus non-empty?

The answer to the first question is 'yes'. We will rely on the results obtained in [24], concerning the GIT-equivalence relation on the space of G-linearized line bundles over a *projective* variety. The main result in *loc. cit.* is the following: consider a normal, projective variety X, acted on by a reductive, linear algebraic group G. One defines $\mathrm{NS}^G(X)$ to be the quotient of $\mathrm{Pic}^G(X)$ by the G-algebraic equivalence relation (see [24], subsection 2.1); it is a finitely generated group, so that $\mathrm{NS}^G(X)_{\mathbb{R}}$ is a finite dimensional vector space over $\mathbb{R}$.

One chooses a W-invariant norm on $\mathcal{X}_*(T)$, and defines the function $\bar{M}:\mathrm{NS}^G(X)\times X\to\mathbb{R}$ similarly as above. It has the property that for all $l\in\mathrm{NS}^G(X)$ holds $X^{\mathrm{ss}}(l)=\{\bar{M}(l,\cdot)\geqslant 0\}$. Moreover $\bar{M}$ uniquely extends to a function on $\mathrm{NS}^G(X)_{\mathbb{R}}\times X$, which is continuous in the first argument (see [7], lemma 3.2.5). For $l\in\mathrm{NS}^G(X)_{\mathbb{R}}$, the l-semi-stable locus is defined as $X^{\mathrm{ss}}(l):=\{\bar{M}(l,\cdot)\geqslant 0\}$.

Definition 3.1. (see [24], section 2) An element $l\in\mathrm{NS}^G(X)_{\mathbb{R}}$ is called *effective* if $X^{\mathrm{ss}}(l)\neq\emptyset$. One says that $l_1,l_2\in\mathrm{NS}^G(X)_{\mathbb{R}}$ are GIT-*equivalent* if $X^{\mathrm{ss}}(l_1)=X^{\mathrm{ss}}(l_2)$.

With these preparations, the main result of [24] reads:

Theorem *Let X be a normal, projective G-variety, and denote by $\mathcal{C}^G(X)$ the set of effective classes in $\mathrm{NS}^G(X)_{\mathbb{R}}$. Then the following hold:*
(i) *$\mathcal{C}^G(X)$ is closed in $\mathrm{NS}^G(X)_{\mathbb{R}}$.*
(ii) *For all $l_o\in\mathcal{C}^G(X)$, $C(l_o):=\{l\in\mathcal{C}^G(X)\mid X^{\mathrm{ss}}(l_o)\subseteq X^{\mathrm{ss}}(l)\}$ is a closed, convex, rational polyhedral cone in $\mathrm{NS}^G(X)_{\mathbb{R}}$.*
(iii) *The cones $C(l)$, $l\in\mathcal{C}^G(X)$ form a fan covering $\mathcal{C}^G(X)$.*
(iv) *The* GIT-*equivalence classes are the relative interiors of these cones.*

The fan constructed in this way is called in [24] the GIT-fan, and will be denoted by $\Delta^G(X)$. We let $\Delta^G(X)_{\mathbb{Q}}:=\Delta^G(X)\cap\mathrm{NS}^G(X)_{\mathbb{Q}}$ and notice that, since the cones of $\Delta^G(X)$ are rational, $\Delta^G(X)$ is obtained from the fan $\Delta^G(X)_{\mathbb{Q}}$ by extending the coefficients from $\mathbb{Q}$ to $\mathbb{R}$.

We are going to adapt this result, holding for projective varieties, to our setting where we deal with actions on affine spaces. For $\chi\in\mathcal{X}^*(G)$, we define $\mathbb{V}^{\mathrm{ss}}(G,\chi):=\{M(\chi,\cdot)\geqslant 0\}$, where M is defined in (2.5).

Theorem 3.2. *Let $\rho:G\to\mathrm{Gl}(V)$ be a representation of the reductive group G which has finite kernel, and consider the induced G-action on $\mathbb{P}(K\oplus V)$ defined by* (2.1). *We denote by $\mathbf{e}$ the trivial character of G. Then the following statements hold:*
(i) *$\mathbb{R}_{\geqslant 0}(1,\mathbf{e})$ is a ray in $\Delta^G\big(\mathbb{P}(K\oplus V)\big)$.*
(ii) *The* GIT-*equivalence classes in $\mathcal{X}^*(G)_{\mathbb{R}}$ corresponding to the G-action on $\mathbb{V}$ are the relative interiors of the cones of the fan*

$$\Delta^G(\mathbb{V}):=\mathrm{star}\big(\mathbb{R}_{\geqslant 0}(1,\mathbf{e})\big)\big/\mathbb{R}(1,\mathbf{e}).$$

We will prove theorem 3.2 in two steps: first we prove the existence of the fan structure on the set of effective characters in $\mathcal{X}^*(G)_{\mathbb{Q}}$; this part uses the results obtained in the previous section. Secondly, we prove that the induced fan in $\mathcal{X}^*(G)_{\mathbb{R}}$, obtained by extending the coefficients from $\mathbb{Q}$ to $\mathbb{R}$, parameterizes the GIT-equivalence classes in $\mathcal{X}^*(G)_{\mathbb{R}}$.

Proof. (theorem 3.2 with $\mathbb{Q}$ coefficients) (i) A direct computation yields $\mathbb{P}(K\oplus V)^{\mathrm{ss}}\big(\Sigma_{1,\mathbf{e}}\big)=\mathbb{V}\cup[\mathbb{V}\setminus\pi^{-1}(\hat{0})]/G_m$, so that $(1,\mathbf{e})$ is an effective class in $\mathrm{NS}^G\big(\mathbb{P}(K\oplus V)\big)=\mathrm{Pic}^G\big(\mathbb{P}(K\oplus V)\big)\cong\mathbb{Z}\oplus\mathcal{X}^*(G)$. From the third part of the theorem above, we deduce that $(1,\mathbf{e})$ is in the relative interior of a cone in $\Delta^G\big(\mathbb{P}(K\oplus V)\big)_{\mathbb{Q}}$.

We claim that $\mathbb{Q}_{\geqslant 0}(1,\mathbf{e})$ is actually a ray in the GIT-fan. Assume the contrary, that $(1,\mathbf{e})$ is in the relative interior of a cone $\tau_0 \in \Delta^G\big(\mathbb{P}(K\oplus V)\big)_{\mathbb{Q}}$; since τ_0 is rational, we find an integral point $l=(d,\chi)$ on its boundary. Then [24], proposition 8, implies that

$$\mathbb{V}\subset \mathbb{P}(K\oplus V)^{\mathrm{ss}}\big(\Sigma_{1,\mathbf{e}}\big)\subsetneq \mathbb{P}(K\oplus V)^{\mathrm{ss}}\big(\Sigma_{d,\chi}\big).$$

In particular, $[1,0]\in \mathbb{P}(K\oplus V)^{\mathrm{ss}}\big(\Sigma_{d,\chi}\big)$, so that there is a homogeneous polynomial F of positive degree, such that

$$F(1,0)\neq 0 \text{ and } F(a,\rho(g)v)=\chi^N(g)\cdot F(a,v),\ \text{for all } (a,v)\in \mathbb{A}^1_K\times\mathbb{V}.$$

Restricting to $a=1$, we obtain $f\in K[\mathbb{V}]^G_{\chi^N}$ such that $f(0)\neq 0$. We deduce that

$$f(0)=f(\rho(g)0)=\chi^N(g)\cdot f(0)\quad\Longrightarrow\quad \chi=\mathbf{e}.$$

It follows that $l=(d,\mathbf{e})$ is collinear with $(1,\mathbf{e})$, which contradicts the choice of l.

(ii) First, consider $\chi\in\mathcal{X}^*(G)$ such that $\mathbb{V}^{\mathrm{ss}}(G,\chi)\neq\emptyset$. For c,d appropriate

$$\mathbb{P}(K\oplus V)^{\mathrm{ss}}\big(\Sigma_{d,\chi^c}\big)\overset{2.3}{=}\mathbb{V}^{\mathrm{ss}}(G,\chi)\cup\big[\mathbb{V}^{\mathrm{ss}}(G,\chi)\setminus\pi^{-1}(\hat{0})\big]/G_m\subseteq \mathbb{P}(K\oplus V)^{\mathrm{ss}}\big(\Sigma_{1,\mathbf{e}}\big),$$

so that, on one hand, (d,χ^c) is effective and therefore contained in the relative interior of a cone τ of $\Delta^G\big(\mathbb{P}(K\oplus V)\big)$; on the other hand, [24], proposition 8, implies that $(1,\mathbf{e})$ belongs to τ. All together shows that $\tau\in\mathrm{star}\big(\mathbb{Q}_{\geqslant 0}(1,\mathbf{e})\big)$.

Using proposition 2.3 again, we deduce that for two GIT-equivalent characters $\chi_1,\chi_2\in\mathcal{X}^*(G)$, that is $\mathbb{V}^{\mathrm{ss}}(G,\chi_1)=\mathbb{V}^{\mathrm{ss}}(G,\chi_2)$, the linearizations Σ_{d,χ_1^c} and Σ_{d,χ_2^c} are still GIT-equivalent, for suitable c,d. Therefore (d,χ_1^c) and (d,χ_2^c) are in the relative interior of the same cone of the GIT-fan of $\mathbb{P}(K\oplus V)$.

Conversely, consider $\tau\in\mathrm{star}\big(\mathbb{Q}_{\geqslant 0}(1,\mathbf{e})\big)$ and an integral point (d,χ) in its relative interior. Lemma 2.2 implies that $\emptyset\neq\mathbb{V}\cap\mathbb{P}(K\oplus V)^{\mathrm{ss}}\big(\Sigma_{d,\chi}\big)\subset\mathbb{V}^{\mathrm{ss}}(G,\chi)$, and therefore χ is effective. Since $(1,\mathbf{e})\in\tau$, $(d+n,\chi)$ is in the relative interior of τ for all $n\geqslant 0$, and therefore the (d,χ)- and $(d+n,\chi)$-semi-stable loci coincide. On the other hand,

$$\mathbb{P}(K\oplus V)^{\mathrm{ss}}\big(\Sigma_{d_1,\chi^{c_1}}\big)=\mathbb{V}^{\mathrm{ss}}(G,\chi)\cup\big[\mathbb{V}^{\mathrm{ss}}(G,\chi)\setminus\pi^{-1}(\hat{0})\big]/G_m$$

for some appropriate c_1, and for all d_1 sufficiently large. Choosing d_1 and n in such a way that $(d+n,\chi)$ and (d_1,χ^{c_1}) are collinear, we deduce that $\mathbb{P}(K\oplus V)^{\mathrm{ss}}\big(\Sigma_{d,\chi}\big)$ is still given by the right-hand-side of the equality above.

Consider now $\tau\in\mathrm{star}\big(\mathbb{Q}_{\geqslant 0}(1,0)\big)$, and two integral points $(d_1,\chi_1),(d_2,\chi_2)$ in its relative interior. Then a similar argument shows that $\mathbb{V}^{\mathrm{ss}}(G,\chi_1)=\mathbb{V}^{\mathrm{ss}}(G,\chi_2)$, that is χ_1 and χ_2 are GIT-equivalent. □

For extending the coefficients from $\mathbb{Q}$ to $\mathbb{R}$, we adapt certain intermediate results in [24] to our context. We recall that $T\subset G$ is a maximal torus of G; for $g\in G$, we denote $T^g:=gTg^{-1}$. The T-module V decomposes into the direct sum of its weight subspaces $V=\oplus_{a\in\Phi}V_a$, and we denote $\{\eta_a\}_{a\in\Phi}$ the corresponding weights. Then $\mathbb{V}=\times_{a\in\Phi}\mathbb{V}_a$, and T acts on $\mathbb{V}_a$ by the character η_a. For $x\in\mathbb{V}$, we write $(x_a)_{a\in\Phi(x)}$ its non-zero coordinates with respect to this decomposition.

Definition 3.3. We define the *stability set* of a point $x\in\mathbb{V}$ to be

$$\Omega(x):=\{\chi\in\mathcal{X}^*(G)_{\mathbb{R}}\mid M(\chi,x)\geqslant 0\}.$$

Lemma 3.4. (i) *For all $x \in \mathbb{V}$, the stability set $\Omega(x) \subset \mathcal{X}^*(G)_{\mathbb{R}}$ is a closed, convex, rational polyhedral cone.*
(ii) *There are only finitely many stability sets.*

Proof. (i) Let $x \in \mathbb{V}$: $\Omega(x)$ is closed since $M(\cdot, x)$ is continuous on $\mathcal{X}^*(G)_{\mathbb{R}}$; $\Omega(x)$ is convex since $M(\chi_1+\chi_2, x) \geqslant M(\chi_1, x) + M(\chi_2, x)$, for all $\chi_1, \chi_2 \in \mathcal{X}^*(G)_{\mathbb{R}}$.

We prove now the rationality property. Define the set $\mathcal{L}_x := \{\lambda \in \mathcal{X}_*(G) \mid m(x,\lambda) \geqslant 0\}$, and notice that

$$\mathcal{L}_x = \bigcup_{g\in G} \mathcal{X}_*(T^g) \cap \mathcal{L}_x = \bigcup_{g\in G} \mathrm{Ad}_g\big(\mathcal{X}_*(T)\cap \mathcal{L}_{\Sigma(g^{-1},x)}\big).$$

Moreover, for $g \in G$, $\mathcal{X}_*(T)\cap\mathcal{L}_{\Sigma(g^{-1},x)} \subset \mathcal{X}_*(T)$ is convex (if we consider $\mathbb{Q}$-coefficients), and

$$\begin{aligned}\mathcal{X}_*(T)\cap\mathcal{L}_{\Sigma(g^{-1},x)} &= \{\lambda \in \mathcal{X}_*(T) \mid m\big(\Sigma(g^{-1},x),\lambda\big) \geqslant 0\}\\ &= \big\{\lambda\in\mathcal{X}_*(T)\,\big|\, \min_{a\in\Phi(\Sigma(g^{-1},x))}\langle \eta_a,\lambda\rangle \geqslant 0\big\}.\end{aligned}$$

Since $\Phi(\Sigma(g^{-1},x)) \subset \Phi$ for all $g \in G$, only a finite number of such sets appear as $g \in G$ varies; let $\Gamma_x \subset G$ be a set of representatives. We deduce that

$$\begin{aligned}M(\chi,x) &= \inf_{\lambda\in\mathcal{L}_x}\frac{\langle\chi,\lambda\rangle}{|\lambda|} = \inf_{g\in G}\inf\left\{\frac{\langle\chi,\lambda\rangle}{|\lambda|}\,\middle|\,\lambda\in\mathrm{Ad}_g(\mathcal{X}_*(T)\cap\mathcal{L}_{\Sigma(g^{-1},x)})\right\}\\ &= \inf_{g\in G}\inf\left\{\frac{\langle\chi,\lambda\rangle}{|\lambda|}\,\middle|\,\lambda\in\mathcal{X}_*(T)\cap\mathcal{L}_{\Sigma(g^{-1},x)}\right\} = \min_{g\in\Gamma_x}\inf\left\{\frac{\langle\chi,\lambda\rangle}{|\lambda|}\,\middle|\,\lambda\in\mathcal{X}_*(T)\cap\mathcal{L}_{\Sigma(g^{-1},x)}\right\}\end{aligned}$$

and therefore

$$\begin{aligned}\chi\in\Omega(x) &\iff \inf\left\{\frac{\langle\chi,\lambda\rangle}{|\lambda|}\,\middle|\,\lambda\in\mathcal{X}_*(T)\cap\mathcal{L}_{\Sigma(g^{-1},x)}\right\}\geqslant 0,\ \forall g\in\Gamma_x\\ &\iff \chi\in\bigcap_{g\in\Gamma_x}\big(\mathcal{X}_*(T)\cap\mathcal{L}_{\Sigma(g^{-1},x)}\big)^\vee.\end{aligned}$$

This is a finite intersection of half-spaces defined by rational equations, so that $\Omega(x)$ is a rational, polyhedral cone.

(ii) Clearly, any stability set is the union of the GIT classes contained in it. There is a finite number of stability sets of the form $\Omega(x)\cap\mathcal{X}^*(G)_{\mathbb{Q}}$: indeed, the proof of theorem 3.2 with $\mathbb{Q}$ coefficients implies that

$$\Omega(x)\cap\mathcal{X}^*(G)_{\mathbb{Q}} = \mathrm{pr}_{\mathcal{X}^*(G)_{\mathbb{Q}}}\big[\Omega([1,x])\cap\text{Support of star}\big(\mathbb{Q}_{\geqslant 0}(1,\mathbf{e})\big)\big],$$

which is clearly finite. By the previous step, $\Omega(x) = \overline{\Omega(x)\cap\mathcal{X}^*(G)_{\mathbb{Q}}}$. □

Lemma 3.5. (i) *For a subset $U \subset \mathbb{V}$, we define $C(U) := \{l \in \mathcal{X}^*(G)_{\mathbb{R}} \mid U \subseteq \mathbb{V}^{\mathrm{ss}}(G,l)\}$. Then $C(U)$ is a closed, convex, rational cone in $\mathcal{X}^*(G)_{\mathbb{R}}$.*
(ii) *For any $\chi_0 \in \mathcal{X}^*(G)_{\mathbb{Q}}$ holds $\mathbb{V}^{\mathrm{ss}}(G,\chi) = \mathbb{V}^{\mathrm{ss}}(G,\chi_0)$, $\quad\forall\chi\in\text{rel.int.}\,C\big(\mathbb{V}^{\mathrm{ss}}(G,\chi_0)\big)$.*

Proof. (i) We notice that $C(U) = \bigcap_{x\in U}\Omega(x)$; by the previous lemma, this is a finite intersection of closed, convex, rational cones, so that $C(U)$ is the same.

(ii) As $\chi\in C\big(\mathbb{V}^{\mathrm{ss}}(G,\chi_0)\big)$, it follows $\mathbb{V}^{\mathrm{ss}}(G,\chi_0)\subset\mathbb{V}^{\mathrm{ss}}(G,\chi)$, and therefore $C\big(\mathbb{V}^{\mathrm{ss}}(G,\chi)\big)\subset C\big(\mathbb{V}^{\mathrm{ss}}(G,\chi_0)\big)$. We deduce that $\chi\in C\big(\mathbb{V}^{\mathrm{ss}}(G,\chi)\big)\cap\text{rel.int.}\,C\big(\mathbb{V}^{\mathrm{ss}}(G,\chi_0)\big)$, and using the first part of the lemma we conclude that there is a *rational* point χ' in the intersection above. Then $\mathbb{V}^{\mathrm{ss}}(G,\chi)\subset\mathbb{V}^{\mathrm{ss}}(G,\chi')$ since $\chi'\in C\big(\mathbb{V}^{\mathrm{ss}}(G,\chi)\big)$, and also $C\big(\mathbb{V}^{\mathrm{ss}}(G,\chi')\big) = C\big(\mathbb{V}^{\mathrm{ss}}(G,\chi_0)\big)$ by theorem 3.2 with rational coefficients. It follows that $\mathbb{V}^{\mathrm{ss}}(G,\chi)\subset\mathbb{V}^{\mathrm{ss}}(G,\chi_0)$. □

Proof. (theorem 3.2 with $\mathbb{R}$ coefficients) We have already proved the statement for rational coefficients, and we know from [24] that the GIT classes in $\Delta^G\big(\mathbb{P}(K\oplus V)\big)$ build a fan structure. Combining these facts with lemma 3.5(ii), we deduce that it remains to prove the following statement: any $\chi\in\mathcal{X}^*(G)_{\mathbb{R}}$ is GIT-equivalent to some rational character $\chi_0\in\mathcal{X}^*(G)_{\mathbb{Q}}$.

Let $\chi\in\mathcal{X}^*(G)_{\mathbb{R}}$ such that $\mathbb{V}^{\rm ss}(G,\chi)\neq\emptyset$. We know from lemma 3.5 that $C\big(\mathbb{V}^{\rm ss}(G,\chi)\big)\subset\mathcal{X}^*(G)_{\mathbb{R}}$ is a rational polyhedral cone. Therefore we find a sequence $\{\chi_n\}_n\subset\mathcal{X}^*(G)_{\mathbb{Q}}$ such that $\chi_n\to\chi$. Since the number of stability sets for rational classes is finite, we may assume that $\mathbb{V}^{\rm ss}(G,\chi_n)=:U$ is independent of n. Now, $\chi_n\in C\big(\mathbb{V}^{\rm ss}(G,\chi_n)\big)=C(U)$ for all n; since χ_n converges to χ and $C(U)$ is closed, we deduce that $\chi\in C(U)$.

There is a unique face $\tau\subset C(U)$, possibly equal to $C(U)$, such that, on one hand, $\chi\in$ rel.int. τ; on the other hand, applying again theorem 3.2 with rational coefficients, we deduce that $\tau=C\big(\mathbb{V}^{\rm ss}(G,\chi_0)\big)$, for any rational point $\chi_0\in$ rel.int. τ. Applying lemma 3.5, we deduce that χ is GIT-equivalent to χ_0. □

Now we describe those characters of G for which the semi-stable locus is non-empty. Obviously the set of effective characters is convex cone in $\mathcal{X}^*(G)_{\mathbb{R}}$, and the natural question which raises is what this cone looks like. Secondly, we wish to characterize the characters $\chi\in\mathcal{X}^*(G)_{\mathbb{R}}$ for which $\mathbb{V}^{\rm ss}(G,\chi)\neq\mathbb{V}^{\rm s}_{(0)}(G,\chi)$. In the terminology of [7], one says that these characters *belong to a wall*; on the other hand, in [24] a wall is by definition a cone of codimension one in $\Delta^G(V)$. Here we will adopt the former definition, but we stress that in this case there may exist walls of codimension zero in $\Delta^G(\mathbb{V})$.

Definition 3.6. A character $l\in\mathcal{X}^*(G)_{\mathbb{R}}$ is *effective* if $\mathbb{V}^{\rm ss}(G,l)\neq\emptyset$. A cone $\tau\in\Delta^G(\mathbb{V})$ is a *wall* if $\mathbb{V}^{\rm ss}(G,l)\neq\mathbb{V}^{\rm s}_{(0)}(G,l)$ for all $l\in$ rel.int. τ.

We start by addressing these issues in the abelian case.

Lemma 3.7. *Let T be a torus, and $\rho:T\to{\rm Gl}(V)$ a representation with weights $\Phi\subset\mathcal{X}^*(T)$.*
(i) *The effective cone is* $\mathcal{C}^T(\mathbb{V})=\displaystyle\sum_{\eta\in\Phi}\mathbb{R}_{\geqslant0}\eta$.
(ii) *The walls are precisely the cones*

$$\tau_{\Phi'}:=\sum_{\eta\in\Phi'}\mathbb{R}_{\geqslant0}\eta,\ \textit{where } \Phi'\subset\Phi \textit{ is such that } {\rm codim}_{\mathcal{X}^*(T)_{\mathbb{R}}}\langle\eta;\eta\in\Phi'\rangle=1.$$

Proof. (i) Since the GIT-fan is rational, we may restrict ourselves to integral coefficients. Consider $\chi=\sum_{\eta\in\Phi(\chi)}k_\eta\eta$, with $\Phi(\chi)\subset\Phi$ and $k_\eta>0$. For each $\eta\in\Phi(\chi)$ we choose a linear function f_η on the weight space $\mathbb{V}_\eta\hookrightarrow\mathbb{V}$. Then $f:=\prod_{\eta\in\Phi(\chi)}f_\eta^{k_\eta}\in K[\mathbb{V}]$ is χ-invariant, and not identically zero. More precisely, f does not vanish at general points of the form $x=(x_\eta)_\eta\in\mathbb{V}$, with $x_\eta=0$ for $\eta\notin\Phi(\chi)$.

Conversely, let $\chi\in\mathcal{X}^*(T)$ be effective, and choose $x=(x_\eta)_\eta\in\mathbb{V}^{\rm ss}(T,\chi)\setminus\{0\}$; define $\Phi(x):=\{\eta\mid x_\eta\neq0\}$. Using proposition 1.2, we deduce that x is χ-semi-stable if and only if:

(3.1)
$$\begin{aligned}&\Big[\forall\,\lambda\in\{\langle\chi,\cdot\rangle<0\}\Rightarrow\min_{\eta\in\Phi(x)}\{\langle\eta,\lambda\rangle\}<0\Big]\iff\{\langle\chi,\cdot\rangle<0\}\cap\Big(\textstyle\sum_{\eta\in\Phi(x)}\mathbb{Z}_{\geqslant0}\eta\Big)^\vee=\emptyset\\&\iff\Big(\textstyle\sum_{\eta\in\Phi(x)}\mathbb{Z}_{\geqslant0}\eta\Big)^\vee\subset\{\langle\chi,\cdot\rangle\geqslant0\}\iff\chi\in\textstyle\sum_{\eta\in\Phi(x)}\mathbb{Z}_{\geqslant0}\eta.\end{aligned}$$

(ii) A character χ is in a wall precisely when there is a point $x=(x_\eta)_\eta\in\mathbb{V}^{\rm ss}(T,\chi)$ with positive dimensional stabilizer. Define $\Phi(x)$ as above. The relation (3.1) implies that

$\chi \in \sum_{\eta\in\Phi(x)} \mathbb{Z}_{\geqslant 0}\eta$. Also, there is a 1-PS λ contained in the stabilizer of x, which means that $\langle\eta,\lambda\rangle = 0$ for all $\eta \in \Phi(x)$. Therefore the $\{\eta\}_{\eta\in\Phi(x)}$'s does not span $\mathcal{X}^*(T)_{\mathbb{R}}$. Conversely, we observe that the point constructed above has positive dimensional stabilizer. □

We consider now the general case: let $\rho : G \to \mathrm{Gl}(V)$ be a representation of a reductive group. We let Φ be the weights of its maximal torus T, and $\mathcal{R}$ be the roots of $\mathfrak{g}$ with respect to T. Let $\mathcal{M}$ be the (finite) set of couples $(\mathcal{A},\mathcal{E}) \subset \mathcal{R}\times\Phi$ defined by the following properties:

$$(3.2)\qquad \begin{array}{ll} \text{(i)} & \exists\,\lambda \in \mathcal{X}_*(T) \text{ s.t. } \begin{array}{l}\mathcal{A} = \{\alpha\in\mathcal{R} \mid \langle\alpha,\lambda\rangle < 0\} = \{\lambda<0\}\cap\mathcal{R}\\ \mathcal{E} = \{\eta\in\Phi \mid \langle\eta,\lambda\rangle < 0\} = \{\lambda<0\}\cap\Phi\end{array}\\ \text{(ii)} & \#\mathcal{A} = \sum\limits_{\eta\in\mathcal{E}} \dim V_\eta. \end{array}$$

For identifying these pairs one proceeds as follows: first, one determines the points of $\mathcal{R}\cup\Phi$ which define hyperplanes $\mathcal{H} \subset \mathcal{X}_*(T)_{\mathbb{Q}}$, and considers the pairs $(\mathcal{A}',\mathcal{E}')$ such that $\mathcal{A}'\cup\mathcal{E}'$ is contained in one of the (closed) half-spaces determined by $\mathcal{H}$; secondly, for the pairs $(\mathcal{A}',\mathcal{E}')$ obtained in this way, one checks the equality (ii) above.

Proposition 3.8. *The cone of effective characters equals*

$$\mathcal{C}^G(\mathbb{V}) = -\bigcap_{(\mathcal{A},\mathcal{E})\in\mathcal{M}} C(\mathcal{A},\mathcal{E}), \quad \textit{where} \quad C(\mathcal{A},\mathcal{E}) := \sum_{\alpha\in\mathcal{A}} \mathbb{R}_{\geqslant 0}\alpha + \sum_{\eta\in\mathcal{E}} \mathbb{R}_{\geqslant 0}\eta.$$

In the case $K[\mathbb{V}] = K$, $\mathcal{C}^G(\mathbb{V})$ is strictly convex.

Proof. Consider $\chi \in \mathcal{X}^*(G)$: it is not effective if and only if

$$\mathbb{V}^{\mathrm{us}}(G,\chi) = \mathbb{V} \iff \exists\,\lambda\in\mathcal{X}_*(T) \text{ s.t. } \langle\chi,\lambda\rangle < 0 \text{ and } G\cdot\mathbb{E}(\lambda) = \mathbb{V}$$

The last equality holds if and only if $\dim\big(G\cdot\mathbb{E}(\lambda)\big) = \dim\mathbb{V}$. For suitably chosen λ holds $\mathrm{Stab}_G\mathbb{E}(\lambda) = P(\lambda)$, *cf.* [15], proposition 4.5, and therefore

$$\dim\big(G\cdot\mathbb{E}(\lambda)\big) = \#\{\alpha\in\mathcal{R}\,|\,\langle\alpha,\lambda\rangle<0\} + \dim\mathbb{E}(\lambda) = \#\{\alpha\in\mathcal{R}\,|\,\langle\alpha,\lambda\rangle<0\} + \sum_{\{\eta\in\Phi|\langle\eta,\lambda\rangle\geqslant 0\}} \dim\mathbb{V}_\eta.$$

The condition above becomes $\#\{\alpha\in\mathcal{R}\,|\,\langle\alpha,\lambda\rangle<0\} = \sum\limits_{\{\eta\in\Phi|\langle\eta,\lambda\rangle<0\}} \dim\mathbb{V}_\eta$, and we obtain a pair $(\mathcal{A},\mathcal{E})$ as in (3.2). One can now turn around the reasoning, and say that the character χ is not effective if and only if there is $(\mathcal{A},\mathcal{E})\in\mathcal{M}$ such that

$$\exists\,\lambda\in\mathcal{X}_*(T) \text{ with } \langle\chi,-\lambda\rangle > 0 \text{ and } -\lambda\in\mathrm{int.}\,C(\mathcal{A},\mathcal{E})^\vee,$$

which means that $\chi \in \bigcup\limits_{\mu\in\mathrm{int.}C(\mathcal{A},\mathcal{E})^\vee} \{\mu > 0\} = \mathcal{X}_*(T)_{\mathbb{Q}} \setminus \big(-C(\mathcal{A},\mathcal{E})\big)$. The cone of effective characters is the complementary of the union of these sets. □

Proposition 3.9. *Let G be a reductive group and $\rho : G \to \mathrm{Gl}(V)$ be a representation with finite kernel. Assume that $\mathcal{X}^*(G)_{\mathbb{R}}$ is not contained in the linear span of any T-wall. Then the walls in $\Delta^G(\mathbb{V})$ are of the form $\mathcal{X}^*(G)_{\mathbb{R}}\cap\tau_{\Phi'}$, with $\Phi'\subset\Phi$ as above. In particular, in this case,* all *walls have codimension one in $\Delta^G(\mathbb{V})$.*

Proof. Since the induced representation $\rho_T : T \to \mathrm{Gl}(V)$ has finite kernel, the weights Φ span $\mathcal{X}^*(T)_{\mathbb{R}}$. The Hilbert-Mumford criterion implies that any wall in $\Delta^G(\mathbb{V})$ is contained in the intersection of $\mathcal{X}^*(G)_{\mathbb{R}}$ with a wall in $\Delta^T(\mathbb{V})$. Our assumption together with the fact that the walls for the T-action on $\mathbb{V}$ have codimension one in $\mathcal{X}^*(T)$ imply that for any wall $\tau\in\Delta^T(\mathbb{V})$, the intersection $\mathcal{X}^*(G)_{\mathbb{R}}\cap\tau\subset\Delta^G(\mathbb{V})$ has codimension one. By [7], proposition

3.3.20, (which, by theorem 3.2, is still valid in our affine setting), every interior wall in $\Delta^G(\mathbb{V})$ is contained in a wall of codimension *at most* one, and deduce that all the interior walls in $\Delta^G(\mathbb{V})$ have the form $\mathcal{X}^*(G)_{\mathbb{R}} \cap \tau$, with $\tau \in \Delta^T(\mathbb{V})$ a wall.

This argument leaves uncovered the exterior cones of $\Delta^G(\mathbb{V})$, but these are obviously walls in the sense of 3.6, and they have (by definition) codimension one. □

Actually, the very same argument as in [7], proposition 3.2.8, shows that if $l \in \mathcal{C}^G(\mathbb{V})$ belongs to an exterior cone of $\Delta^G(\mathbb{V})$, then $\mathbb{V}^{\rm s}_{(0)}(G,l) = \emptyset$.

The disadvantage of this proposition is that of involving the maximal torus of G, which is typically much larger than its centre. For explicit computations, the observations below can be quite useful.

Remark 3.10. (i) $\mathcal{C}^G(\mathbb{V}) \subset \mathcal{C}^{Z(G)^\circ}(\mathbb{V})$;
(ii) A character $\chi \in \mathcal{X}^*(G)$ belongs to a wall if and only if the χ-poly-stable locus is non-empty (*i.e.* there is $x \in \mathbb{V}^{\rm ss}(G,\chi)$ whose stabilizer is positive dimensional, and its orbit is closed in $\mathbb{V}^{\rm ss}(G,\chi)$). In this case $\mathrm{Stab}_G(x)$ is reductive, and for any $\lambda \in \mathcal{X}_*\big(\mathrm{Stab}_G(x)\big)$ holds:
(ii$_{\rm a}$) $x \in \mathbb{V}_{\lambda,0} := \{y \in \mathbb{V} \mid \lambda \subset \mathrm{Stab}_G(y)\}$;
(ii$_{\rm b}$) $\mathbb{V}_{\lambda,0} \hookrightarrow \mathbb{V}$ is a linear subspace, invariant under the parabolic subgroup $P(\lambda) \subset G$;
(iii) Let $L(\lambda) := Z_{P(\lambda)}(\lambda)$ denote the centralizer of λ in $P(\lambda)$; it also coincides with the Levi component of $P(\lambda)$. Then $\chi|_{\mathrm{Ker}\left(\rho|_{L(\lambda)}:L(\lambda)\to \mathrm{Gl}(V_{\lambda,0})\right)^\circ} \equiv 1$. This condition should be viewed as *a restriction* for the character χ lying in the wall since the parabolic subgroups of G containing a fixed Borel subgroup of G are described by combinatorial data.

All statements above are clear except maybe the last one: since x is χ-semi-stable, we deduce from the very definition that χ is trivial on $\mathrm{Stab}_G(x)^\circ$, and this one contains $\mathrm{Ker}(\rho|_{L(\lambda)})^\circ$.

Example 3.11. We wish to illustrate our remark in the case of quiver factorization problems, where we recover the semi-stability criterion in [20], proposition 3.2.

Consider a quiver $Q = (Q_0, Q_1, h, t)$, where h, t are the head and the tail maps respectively, and fix a dimension vector $(\alpha_q)_{q\in Q_0}$. We let $V := \bigoplus_{a\in Q_1} \mathrm{Hom}\big(U_{h(a)}, U_{t(a)}\big)$ be a representation space for Q, with $\dim U_q = \alpha_q$ for all $q \in Q_0$. We consider further the quiver representation $\rho : G = \prod_{q\in Q_0} \mathrm{Gl}(U_q) \to \mathrm{Gl}(V)$, and we wish to determine the *possible* walls for this action. In this case proposition 3.9 is very ineffective because the maximal torus is large, while the dimension of the centre equals $|Q_0|$.

Assume that the character $\chi_\theta := \prod_{q\in Q_0} \det_q^{\theta_q}$ is in a wall, and let $\lambda = (\lambda_q)_{q\in Q_0}$ be the 1-PS as above. For each $q \in Q_0$, $U_q = \oplus_j U_{q,j}$ decomposes into the direct sum of the weight spaces corresponding to λ_q, and we find that

$$V_{\lambda,0} = \bigoplus_{a\in Q_1}\Big(\bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}\big(U_{s(a),j}, U_{t(a),j}\big)\Big) = \bigoplus_{j\in\mathbb{Z}}\Big(\bigoplus_{a\in Q_1} \mathrm{Hom}\big(U_{s(a),j}, U_{t(a),j}\big)\Big), \text{ and}$$
$$L(\lambda) = \prod_{q\in Q_0}\Big(\prod_{j\in\mathbb{Z}} \mathrm{Gl}(U_{q,j})\Big) = \prod_{j\in\mathbb{Z}}\Big(\prod_{q\in Q_0} \mathrm{Gl}(U_{q,j})\Big).$$

For $j \in \mathbb{Z}$ holds $\prod\limits_{q\in Q_0,\, \dim U_{q,j}>0} G_m \subset \big(\mathrm{Ker}\rho|_{L(\lambda)}\big)^\circ$, and therefore $\sum\limits_{q\in Q_0} \theta_q \cdot \dim U_{q,j} = 0$. The conclusion is that the *potential* walls in $\mathbb{R}^{|Q_0|}$ have the form

$$\sum_{q\in Q_0} \beta_q\theta_q = 0, \text{ where } 0 \le \beta_q \le \alpha_q \ \forall\, q \in Q_0.$$

In particular, for quiver factorization problems, all walls have codimension one.

More generally, for a subset $S \subset Q_0$ one defines $G_S := \prod_{q\in S} \mathrm{Gl}(U_q)$ and asks the equations of the walls for the induced representation $\rho_S : G_S \to \mathrm{Gl}(V)$. Replacing $\theta_q = 0$ for $q \notin S$, we get that the potential walls are

$$\sum_{q\in S} \beta_q\theta_q = 0, \text{ where } 0 \le \beta_q \le \alpha_q \;\forall\, q \in S. \tag{3.3}$$

For several explicit examples of quiver factorization problems one may consult [16].

4. The Chow ring of the quotients

Let $\rho : G \to \mathrm{Gl}(V)$ be a representation with finite kernel. Our goal in this section is to compute the Chow ring of the invariant quotients of $\mathbb{V}$, corresponding to characters of G. The main tool for this computation is the equivariant intersection theory developed in [8]. For shorthand, we denote by A^G_* the equivariant Chow ring of a point.

First we recall some results from [6]: the *discriminant* $\Delta_G \in \mathrm{End}(\mathrm{Sym}^\bullet \mathcal{X}^*(T))$ is defined to be the product of the linear forms corresponding to the reflections in the Weyl group. This definition determines it is uniquely, up to sign. Equivalently, for a choice $T \subset B \subset G$ of a Borel subgroup, Δ_G is defined as the product of the positive roots. For $w \in W = N_G(T)/T$, we let $\det(w)$ be the determinant of w, viewed as an endomorphism of $\mathcal{X}^*(T)$. The discriminant has the property that for all $w \in W$, $w(\Delta_G) = \det(w)\Delta_G$. Since $A^*_T \cong \mathrm{Sym}^\bullet \mathcal{X}^*(T)$, we will view Δ_G as an element in $A_T^{\dim G/B}$.

We consider the $(A^*_T)^W$-linear endomorphism $J_G := \sum\limits_{w\in W} \det(w)w$ of A^*_T; according to [6], lemme 4, it has the property that $\wp_G := J_G/\Delta_G$ is still an endomorphism of A^*_T, which takes values in $(A^*_T)^W$. Since $\wp_G : (A^*_T)_{\mathbb{Q}} \to (A^*_T)^W_{\mathbb{Q}}$ is $(A^*_T)^W_{\mathbb{Q}}$-linear, and $\wp_G(\Delta_G) = |W|$, it is an epimorphism. The geometrical meaning of the homomorphism $\wp_G$ is captured in the

Lemma 4.1. *The composition* $A^*_T \cong A^*_B \xrightarrow{\varphi_*} A_G^{*-\dim G/B} \xrightarrow{\varphi^*} A_T^{*-\dim G/B}$ *equals* $\wp_G$.

Proof. The push-forward formula implies that the endomorphism $h := \varphi^*\varphi_*$ is $\varphi^*(A^*_G)$-linear. But $\varphi^* : A^*_G \to (A^*_T)^W$ is injective, its cokernel is $\mathbb{Z}$-torsion, and therefore h is $(A^*_T)^W$-linear. Moreover, h vanishes in degree strictly less than $\dim G/B = \deg \Delta_G$. Applying [6], proposition 1, we obtain that $|W| \cdot \Delta_G \cdot h(u) = h(\Delta_G) \cdot J_G(u) = |W| \cdot J_G(u)$. □

With these preparations, the main result of this section is

Theorem 4.2. *Let* $\chi \in \mathcal{X}^*(G)$ *be a character such that* $\mathbb{V}^{\mathrm{ss}}(G,\chi) = \mathbb{V}^{\mathrm{s}}_{(0)}(G,\chi)$*; in particular* $\mathbb{V}/\!\!/_\chi G$ *is a geometric quotient. Then the Chow ring*

$$A_*\big(\mathbb{V}/\!\!/_\chi G\big)_{\mathbb{Q}} \cong \big(A^T_*\big)^W_{\mathbb{Q}} \Big/ \wp_G\big\langle [\mathbb{E}(\lambda)]_T ; \lambda \in \mathcal{F}(\chi)\big\rangle_{\mathbb{Q}}.$$

This result represents a considerable generalization of [10], theorem 4.4, where one assumes that $\rho(G)$ contains the homotheties of $\mathrm{Gl}(V)$ (which implies that $K[\mathbb{V}]^G = K$), and moreover that G acts freely on $\mathbb{V}^{\mathrm{s}}_{(0)}(G,\chi)$. The computation of $[\mathbb{E}(\lambda)]_T$ is immediate: if $E \hookrightarrow V$ is a T-submodule, then the equivariant class of the linear space $\mathbb{E} := \mathrm{Spec}\big(\mathrm{Sym}^\bullet E^\vee\big)$ is

$$[\mathbb{E}]_T = \prod_a \eta_a^{m_a} \in A^T_*,$$

where $\{\eta_a\}_a$ are the weights of the T-module V/E, and $\{m_a\}_a$ are the corresponding multiplicities.

Lemma 4.3. *Let $P \subset G$ be a parabolic subgroup, $L \subset P$ its Levi component, and consider $\tilde{\eta} \in A^*_T(G/P)^W$. Further, denote by $f^T_* : A^*_T(G/P) \to A^*_T$ the proper push-forward.*
(i) *The fixed point set of G/P under the natural T-action consists of finitely many points:*
$$\big(G/P\big)^T = \big\{wP \mid w \in N_G(T)/N_L(T) = W/W_L\big\}.$$
(ii) *Let $\imath_1 : \{P\} \hookrightarrow G/P$ be the inclusion, and $\eta := \imath_1^*\tilde{\eta}$. Then $f^T_*\big(\langle\tilde{\eta}\rangle^W_{\mathbb{Q}}\big) = \wp_G\langle\eta\rangle_{\mathbb{Q}}$.*

Proof. The first statement is well-known. For the second one, we use the integration formula [9], corollary 1, applied to the homogeneous variety G/P. For $w \in W$ consider the diagram

$$\begin{array}{ccc} \{wP\} & \xhookrightarrow{\ \imath_w\ } & G/P \\ {\scriptstyle f_w}\downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} K & = & \operatorname{Spec} K \end{array}$$

Observe that for $\tilde{\alpha} \in \langle\tilde{\eta}\rangle^W_{\mathbb{Q}}$, $\imath_w^*\tilde{\alpha} = w(\imath_1^*\tilde{\alpha})$; in particular $\imath_1^*\tilde{\alpha} \in A^*_T$ is W_L-invariant. Choosing a set of representatives $\hat{W} \subset W$ for the rest classes in W/W_L, with $1 \in \hat{W}$, the integration formula reads

$$f^T_*(\tilde{\alpha}) = \sum_{\hat{w}\in\hat{W}} (f^T_{\hat{w}})_* \frac{\imath^*_{\hat{w}}\tilde{\alpha}}{\mathsf{e}^T\big(\mathsf{T}_{\hat{w}P}(G/P)\big)} = \sum_{\hat{w}\in\hat{W}} \hat{w}\left(\frac{(f^T_1)_*\imath_1^*\tilde{\alpha}}{(f^T_1)_*\mathsf{e}^T\big(\mathsf{T}_P(G/P)\big)}\right). \tag{4.1}$$

We observe now that the image of $(f^T_1)_*\imath_1^* : A^*_T(G/P)^W_{\mathbb{Q}} \to A^*_T$ equals $(A^*_T)^{W_L}_{\mathbb{Q}}$, and also that $(f^T_1)_*\imath_1^*\tilde{\eta} = \eta$. It follows that $(f^T_1)_*\imath_1^*\langle\tilde{\eta}\rangle^W_{\mathbb{Q}} = \eta\cdot(A^*_T)^{W_L}_{\mathbb{Q}}$.

Secondly, we claim that $(f^T_1)_*\mathsf{e}^T\big(\mathsf{T}_P(G/P)\big)$ equals the quotient Δ_G/Δ_L of the discriminants of G and L respectively. Indeed, if $\mathfrak{g}$, $\mathfrak{p}$ and $\mathfrak{b}$ are the Lie algebras of G, P and B respectively, then $(f^T_1)_*\mathsf{e}^T\big(\mathsf{T}_P(G/P)\big)$ is the product of weights of the T-module $\mathfrak{g}/\mathfrak{p} \cong (\mathfrak{g}/\mathfrak{b})/(\mathfrak{p}/\mathfrak{b})$. Remains to notice that the weights of $\mathfrak{p}/\mathfrak{b}$ are (up to sign) precisely the positive roots of L.

We rewrite equality above as

$$\begin{aligned} f^T_*(\tilde{\alpha}) &= \sum_{\hat{w}\in\hat{W}} \hat{w}\left(\frac{\alpha}{\Delta_G/\Delta_L}\right), \quad \text{with } \alpha := (f^T_1)_*\imath_1(\tilde{\alpha}) \in \eta\cdot(A^*_T)^{W_L}_{\mathbb{Q}} \\ &= \frac{1}{|W_L|}\sum_{w\in W} w\left(\frac{\alpha}{\Delta_G/\Delta_L}\right) = \frac{1}{|W_L|}\wp_G(\Delta_L\cdot\alpha). \end{aligned} \tag{4.2}$$

We are going to compute $\wp_G\langle\eta\rangle_{\mathbb{Q}}$, and compare with this formula. For the same reason as before, $\eta \in (A^*_T)^{W_L}$. For $\eta b \in \langle\eta\rangle_{\mathbb{Q}}$, we obtain

$$\begin{aligned} \wp_G(\eta b) &= \frac{1}{\Delta_G}\sum_{w\in W}\det(w)\cdot w(\eta b) = \frac{1}{\Delta_G}\sum_{\hat{w}\in\hat{W}}\sum_{w\in\hat{w}W_L}\det(w)\cdot w(\eta b) \\ &= \frac{1}{\Delta_G}\sum_{\hat{w}\in\hat{W}}\hat{w}(\eta)\cdot\sum_{w\in\hat{w}W_L}\det(w)\cdot w(b) \\ &= \frac{1}{\Delta_G}\sum_{\hat{w}\in\hat{W}}\det(\hat{w})\cdot\hat{w}(\eta)\cdot\hat{w}\left(\sum_{w\in W_L}\det(w)\cdot w(b)\right) \\ &= \frac{1}{\Delta_G}\sum_{\hat{w}\in\hat{W}}\det(\hat{w})\cdot\hat{w}(\eta)\cdot\hat{w}\big(\Delta_L\cdot\wp_L(b)\big) = \sum_{\hat{w}\in\hat{W}}\hat{w}\left(\frac{\eta\cdot\wp_L(b)}{\Delta_G/\Delta_L}\right). \end{aligned}$$

As $\wp_L : (A_T^*)_{\mathbb{Q}} \to (A_T^*)_{\mathbb{Q}}^{W_L}$ is surjective, $\{\eta \cdot \wp_L(b) \mid b \in (A_T^*)_{\mathbb{Q}}\} = \eta \cdot (A_T^*)_{\mathbb{Q}}^{W_L}$. □

Proposition 4.4. *Let $\rho : G \to \mathrm{Gl}(V)$ be a representation with finite kernel, and $\mathbb{E} \hookrightarrow \mathbb{V}$ a linear subspace such that its stabilizer $P := \mathrm{Stab}_G(\mathbb{E})$ is a parabolic subgroup of G. Then the image of the canonical homomorphism*

$$A_*^G(G \cdot \mathbb{E})_{\mathbb{Q}} \longrightarrow A_*^G(\mathbb{V})_{\mathbb{Q}} \cong (A^G_{*-\dim \mathbb{V}})_{\mathbb{Q}}$$

equals $\wp_G \langle [\mathbb{E}]_T \rangle_{\mathbb{Q}}$.

Proof. (i) The statement makes sense: $G \cdot \mathbb{E}$ is closed in $\mathbb{V}$ since the stabilizer of $\mathbb{E}$ is parabolic. Let $e := \dim \mathbb{E}$, and consider the action of G on the Grassmannian $\mathrm{Grass}(e, \mathbb{V})$ of e-dimensional linear subspaces of $\mathbb{V}$. We denote $\mathscr{T} \to \mathrm{Grass}(e, \mathbb{V})$ the tautological bundle of rank e, and by $\mathrm{O}_{\mathbb{E}} \cong G/P$ the G-orbit of $[\mathbb{E}] \in \mathrm{Grass}(e, \mathbb{V})$. Consider the commutative diagram

$$\begin{array}{ccc} \mathscr{T}|_{\mathrm{O}_{\mathbb{E}}} & \overset{\jmath}{\hookrightarrow} & \mathrm{O}_{\mathbb{E}} \times \mathbb{V} \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{\mathbb{V}}} \\ G \cdot \mathbb{E} & \hookrightarrow & \mathbb{V} \end{array}$$

The homomorphism σ_* is surjective since σ is a projective morphism. We deduce that image of $A_*^G(G \cdot \mathbb{E})_{\mathbb{Q}} \longrightarrow A_*^G(\mathbb{V})_{\mathbb{Q}}$ equals the image of $(\mathrm{pr}_{\mathbb{V}})_*^G \circ \jmath_*^G$. We consider the commutative diagram of homomorphisms between G-equivariant Chow groups

$$\begin{array}{ccccc} A_*^G(\mathscr{T}|_{\mathrm{O}_{\mathbb{E}}}) & \xrightarrow{\jmath_*^G} & A_*^G(\mathrm{O}_{\mathbb{E}} \times \mathbb{V}) & \xrightarrow{(\mathrm{pr}_{\mathbb{V}})_*^G} & A_*^G(\mathbb{V}) \\ \cong\uparrow & & \cong\uparrow & & \cong\uparrow \\ A^G_{*-e}(\mathrm{O}_{\mathbb{E}}) & \xrightarrow{\mathsf{e}^G(\mathscr{Q}|_{\mathrm{O}_{\mathbb{E}}}) \cap} & A^G_{*-\dim \mathbb{V}}(\mathrm{O}_{\mathbb{E}}) & \xrightarrow{f_*^G} & A^G_{*-\dim \mathbb{V}} \end{array}$$

and the excess intersection theorem (see [11], example 6.3.5) implies that the lower left homomorphism is the cap product with the equivariant Euler class of the universal quotient bundle $\mathscr{Q} \to \mathrm{Grass}(e, \mathbb{V})$. By abuse of notation, we will still write $\mathscr{Q} \to \mathrm{O}_{\mathbb{E}}$ for the restriction of $\mathscr{Q}$ to $\mathrm{O}_{\mathbb{E}}$. Let us consider an appropriate open subset U in some representation space of G (see [8], definition-proposition 1) needed for the computation of the equivariant Chow groups. The diagram

$$\begin{array}{ccc} U \times_T \mathrm{O}_{\mathbb{E}} & \xrightarrow{f^T} & U/T \\ {\scriptstyle \varphi_{U,\mathrm{O}_{\mathbb{E}}}}\downarrow & & \downarrow{\scriptstyle \varphi_U} \\ U \times_G \mathrm{O}_{\mathbb{E}} & \xrightarrow{f^G} & U/G \end{array}$$

is Cartesian, the horizontal morphisms are proper and the vertical ones are flat, and therefore the induced diagram

$$\begin{array}{ccc} \left\langle \mathsf{e}^T(\mathscr{Q}) \right\rangle_{\mathbb{Q}} \subset A_*^T(\mathrm{O}_{\mathbb{E}})_{\mathbb{Q}} & \xrightarrow{f_*^T} & (A_*^T)_{\mathbb{Q}} \\ {\scriptstyle \varphi^*}\uparrow & & \uparrow{\scriptstyle \varphi^*} \\ \left\langle \mathsf{e}^G(\mathscr{Q}) \right\rangle_{\mathbb{Q}} \subset A_*^G(\mathrm{O}_{\mathbb{E}})_{\mathbb{Q}} & \xrightarrow{f_*^G} & (A_*^G)_{\mathbb{Q}} \end{array}$$

commutes (see [11], proposition 1.7). Now, we know from [8], proposition 6, that φ^* is injective, and its image consists of the W-invariant elements, that is

$$\varphi^*\big[f_*^G\big(\langle \mathsf{e}^G(\mathscr{Q})\rangle_{\mathbb{Q}}\big)\big] = f_*^T\big(\langle \mathsf{e}^T(\mathscr{Q})\rangle_{\mathbb{Q}}^W\big) \overset{4.3}{=} \wp_G\big(\langle[\mathbb{E}]_T\rangle_{\mathbb{Q}}^W\big).$$

All together shows that the image of $A_*^G(G\cdot\mathbb{E}) \to A^G_{*-\dim\mathbb{V}}$ is $\wp_G\big(\langle[\mathbb{E}]_T\rangle_{\mathbb{Q}}^W\big)$. □

Proof. (of the theorem) We have proved in lemma 1.1 that

$$\mathbb{V}^{\rm s}_{(0)}(G,\chi)/G = \big(\mathbb{V}\setminus\mathbb{V}^{\rm us}(G,\chi)\big)/G = \Bigg(\mathbb{V}\setminus\bigcup_{\lambda\in\mathcal{F}(\chi)} G\cdot\mathbb{E}(\lambda)\Bigg)\Bigg/G,$$

where the $\mathbb{E}(\lambda)$'s are linear subspaces of $\mathbb{V}$. Applying theorem 3 and proposition 5 of [8], we deduce that $A_{*-\dim G}\big(\mathbb{V}^{\rm s}_{(0)}(G,\chi)/G\big) \cong A_*^G\big(\mathbb{V}^{\rm s}_{(0)}(G,\chi)\big)$, and that there is an exact sequence

$$A_*^G\Bigg(\bigcup_{\lambda\in\mathcal{F}(\chi)} G\cdot\mathbb{E}(\lambda)\Bigg) \xrightarrow{\jmath_*^G} A_*^G(\mathbb{V}) \longrightarrow A_*^G\big(\mathbb{V}^{\rm s}_{(0)}(G,\chi)\big) \longrightarrow 0.$$

Further, by the very construction of $\mathcal{F}(\chi)$, the $\{G\cdot\mathbb{E}(\lambda)\}_{\lambda\in\mathcal{F}(\chi)}$ are the irreducible components of the union, and therefore

$$\jmath_*^G\Bigg[A_*^G\Bigg(\bigcup_{\lambda\in\mathcal{F}(\chi)} G\cdot\mathbb{E}(\lambda)\Bigg)\Bigg] = \sum_{\lambda\in\mathcal{F}(\chi)} \jmath_*^G\big[A_*^G\big(G\cdot\mathbb{E}(\lambda)\big)\big].$$

For $\lambda\in\mathcal{F}(\chi)$, $\mathbb{E}(\lambda)$ is stabilized by the parabolic subgroup $P(\lambda)\subset G$. Proposition 4.4 implies that $\jmath_*^G\big[A_*^G\big(G\cdot\mathbb{E}(\lambda)\big)\big]_{\mathbb{Q}} = \wp_G\langle[\mathbb{E}(\lambda)]_T\rangle_{\mathbb{Q}}$. □

Remark 4.5. We have been primarily interested in the Chow ring of the quotients, but the proof shows more: for *any* $\chi\in\mathcal{X}^*(G)$ holds:

$$A_*^G\big(\mathbb{V}^{\rm ss}(G,\chi)\big)_{\mathbb{Q}} \cong \big(A_*^T\big)_{\mathbb{Q}}^W\Big/\wp_G\big\langle[\mathbb{E}(\lambda)]_T;\lambda\in\mathcal{F}(\chi)\big\rangle_{\mathbb{Q}}, \quad\text{and}$$
$$A_*^G\big(\mathbb{V}^{\rm s}_{(0)}(G,\chi)\big)_{\mathbb{Q}} \cong \big(A_*^T\big)_{\mathbb{Q}}^W\Big/\wp_G\big\langle[\mathbb{E}(\lambda)]\,;\,\lambda\in\{\langle\chi,\cdot\rangle\leqslant 0\}\big\rangle_{\mathbb{Q}}.$$

We conclude this section with a remark about the generators of the Chow ring. In [10], section 6, last paragraph, the authors express their belief that the Chow ring of invariant quotients of affine spaces, should be generated by Chern classes of 'naturally given vector bundles'.

Following [26], example 3, page 26, we define the representation ring $\mathcal{R}(G)$ of G to be the ring generated by isomorphism classes of representations of G modulo the ideal generated by $[F]-[F']-[F'']$, where

$$0\longrightarrow F'\longrightarrow F\longrightarrow F''\longrightarrow 0$$

is an exact sequence of G-modules; if $\operatorname{char} K = 0$, reductive groups are linearly reductive, and in this case $F = F'\oplus F''$ as a G-module. The addition in $\mathcal{R}(G)$ is given by direct sum, and the product by tensor product.

The restriction to the maximal torus defines a ring homomorphism $\mathcal{R}(G)\longrightarrow\mathcal{R}(T)^W$, where on the right-hand-side we consider the W-invariant representations of T. By [26], théorème 1.1, page 26, this homomorphism is actually an isomorphism. Since the representation ring of T is isomorphic to the symmetric algebra of its group of characters, we deduce the isomorphism

$$\mathcal{R}(G)_{\mathbb{Q}} \xrightarrow{\cong} \big(\mathrm{Sym}^{\bullet}_{\mathbb{Q}}\, \mathcal{X}^*(T)_{\mathbb{Q}}\big)^W \cong (A^*_T)^W_{\mathbb{Q}} \cong (A^*_G)_{\mathbb{Q}}. \tag{4.3}$$

Moreover, the composition $\mathcal{R}(G)_{\mathbb{Q}} \to (A^*_G)_{\mathbb{Q}}$ is given by the Chern character

$$\mathcal{R}(G) \ni (F,\rho) \longmapsto \mathrm{ch}\big(\mathrm{EG} \times_{\rho} F \to \mathrm{BG}\big) \in A^*_G.$$

By abuse of notation, EG stands here for an appropriate open subset of an affine space, needed to define equivariant classes, and $\mathrm{BG} := \mathrm{EG}/G$.

Proposition 4.6. (i) *Let $\ell := \operatorname{rank} G = \dim T$. There are at most ℓ elements in $\mathcal{R}(G)$ whose Chern classes generate the ring $(A^*_T)^W_{\mathbb{Q}}$.*
(ii) *Consider $\chi \in \mathcal{X}^*(G)$ such that $\mathbb{V}^{\mathrm{ss}}(G,\chi) = \mathbb{V}^{\mathrm{s}}_{(0)}(G,\chi)$. The same statement as in* (i) *holds for the Chow ring of the invariant quotient $\mathbb{V}/\!\!/_{\chi}G$.*

Proof. Since $(A^T_*)^W_{\mathbb{Q}} \to A_*(Y)_{\mathbb{Q}}$ is an epimorphism, it is enough to prove the first claim. Applying [1], section 5.5, théorème 4, to the $\mathbb{Q}$-vector space $\mathcal{X}^*(T)_{\mathbb{Q}}$, we deduce that there are algebraically independent, homogeneous elements $I_1,\dots,I_\ell \in (A^T_*)^W_{\mathbb{Q}}$, such that $(A^T_*)^W_{\mathbb{Q}} = \mathbb{Q}[I_1,\dots,I_\ell]$. The isomorphism (4.3) implies that there are elements $F_1,\dots,F_\ell \in \mathcal{R}(G)$ such that

$$\mathrm{ch}_{\deg I_j}(F_j) = I_j, \quad \forall\, j = 1,\dots,\ell,$$

so that the Chern classes of $F_1,\dots,F_\ell$ generate $(A^T_*)^W_{\mathbb{Q}}$. □

Example 4.7. A particularly comfortable situation arises when $G = \times_{j=1}^{s}\mathrm{Gl}(n_j)$ is a product of linear groups; this happens, for instance, in the case of quiver representations. Then the Chow ring of the corresponding quotients will be generated by the Chern classes of the natural representations of the s factors of G.

5. The cohomology ring of the quotients

In this section we assume that we work over the field of complex numbers, and moreover that the ring of invariants $\mathbb{C}[\mathbb{V}]^G = \mathbb{C}$. This latter condition guarantees that the invariant quotients $\mathbb{V}/\!\!/_{\chi}G$, $\chi \in \mathcal{X}^*(G)$, are projective.

Theorem 5.1. *Let $\chi \in \mathcal{X}^*(G)$ be a character such that $\mathbb{V}^{\mathrm{ss}}(G,\chi) = \mathbb{V}^{\mathrm{s}}_{(0)}(G,\chi)$. Then the cohomology ring*

$$H^*(\mathbb{V}/\!\!/_{\chi}G;\mathbb{Q}) \cong \big(H^*_T\big)^W_{\mathbb{Q}} \Big/ \wp_G\big\langle [\mathbb{E}(\lambda)]_T; \lambda \in \mathcal{F}(\chi)\big\rangle_{\mathbb{Q}}.$$

In particular, the cycle map $\mathrm{cl} : A_*\big(\mathbb{V}/\!\!/_{\chi}G\big)_{\mathbb{Q}} \longrightarrow H^*(\mathbb{V}/\!\!/_{\chi}G;\mathbb{Q})$ is an isomorphism.

Proof. The idea is to use the results of section 2 and to reduce our question to a similar one for actions of reductive groups on projective varieties, for which we know the cohomology of the semi-stable locus.

We have proved in proposition 2.3 that

$$\mathbb{V}^{\mathrm{ss}}(G,\chi) = \mathbb{P}(\mathbb{C} \oplus V)^{\mathrm{ss}}\big(G, \Sigma_{(d,\chi^c)}\big) =: \Omega.$$

For the convenience of the writing, we shall denote $\bar{\mathbb{V}} := \mathbb{P}(\mathbb{C} \oplus V)$.

We claim that the restriction $H^*_G(\mathbb{V}) \to H^*_G(\Omega)$ is surjective: indeed, the diagram

$$\begin{array}{ccc} H^*_G(\bar{\mathbb{V}}) & \xrightarrow{\jmath^*_G} & H^*_G(\mathbb{V}) \\ & \searrow^{\alpha^*} & \downarrow \\ & & H^*_G(\Omega) \end{array}$$

commutes, and we know from [22], chapter 8, section 7, that α^* is surjective. Secondly, we claim that $\jmath^*_G$ is surjective too. The commutative diagram

$$\begin{array}{ccc} \mathrm{BG} \cong \mathrm{EG} \times_G [1,0] & \xhookrightarrow{\jmath_0} & \mathrm{EG} \times_G \bar{\mathbb{V}} \\ & \diagdown\!\!\diagdown & \downarrow \\ & & \mathrm{BG} \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} H^*_G & \longrightarrow & H^*_G(\bar{\mathbb{V}}) \\ & \diagdown\!\!\diagdown^{\mathrm{id}} & \downarrow (\jmath_0)^*_G \\ & & H^*_G \end{array}$$

implies that $(\jmath_0)^*_G$ is surjective. As usual, $\mathrm{EG} \to \mathrm{BG}$ stays for a universal G-bundle. The surjectivity of $\jmath^*_G$ follows now from the commutative triangle

$$\begin{array}{ccc} H^*_G(\bar{\mathbb{V}}) & \xrightarrow{\jmath^*_G} & H^*_G(\mathbb{V}) \\ & \searrow^{(\jmath_0)^*_G} & \downarrow \cong \\ & & H^*_G \end{array} \tag{5.1}$$

corresponding to the inclusions $\{0\} \subset \mathbb{V} \subset \bar{\mathbb{V}}$.

The two claims together imply that

$$\mathrm{Ker}\big(H^*_G(\mathbb{V}) \to H^*_G(\Omega)\big) = \jmath^*_G \mathrm{Ker}\big(H^*_G(\bar{\mathbb{V}}) \to H^*_G(\Omega)\big). \tag{5.2}$$

Now we are going to use the result [3], corollaire 1.1, which expresses the right-hand-side in terms of the W-invariant part of the T-equivariant cohomology of the T-unstable locus. For this, we consider the diagram

$$\begin{array}{ccc} H^*_T(\mathbb{V}) & \xleftarrow{\ \varphi^*\ } & H^*_G(\mathbb{V}) \\ \jmath^*_T \uparrow & & \uparrow \jmath^*_G \\ H^*_T(\bar{\mathbb{V}}) & \xleftarrow[\ \varphi^*\ ]{} & H^*_G(\bar{\mathbb{V}}) \end{array}$$

Applying φ^* to both sides of (5.2), we obtain that

$$\begin{aligned} \varphi^* \mathrm{Ker}\big(H^*_G(\mathbb{V}) \to H^*_G(\Omega)\big) &= (\varphi^* \circ \jmath^*_G) \mathrm{Ker}\big(H^*_G(\bar{\mathbb{V}}) \to H^*_G(\Omega)\big) \\ &= (\jmath^*_T \circ \varphi^*) \mathrm{Ker}\big(H^*_G(\bar{\mathbb{V}}) \to H^*_G(\Omega)\big) \\ &\overset{loc.\,cit.}{=} \jmath^*_T \left[\mathrm{Ker}\Big(H^*_T(\bar{\mathbb{V}}) \to H^*_T\big(\bar{\mathbb{V}}^{\mathrm{ss}}(T, \Sigma_{(d,\chi^c)})\big) \Big)^W \right] \\ &= \left[\jmath^*_T \mathrm{Ker}\Big(H^*_T(\bar{\mathbb{V}}) \to H^*_T\big(\bar{\mathbb{V}}^{\mathrm{ss}}(T, \Sigma_{(d,\chi^c)})\big) \Big) \right]^W. \end{aligned}$$

The last equality holds because the projection onto the W-invariant part is a linear map (a Reynolds type homomorphism), which commutes with the pull-back. Let us point out that in order to apply Brion's result, we must be able to linearize the G-action in $\mathscr{O}_{\bar{\mathbb{V}}}(1)$; according to remark 2.4, this can be achieved after replacing G with a suitable finite cover (see 2.4). This modification does not affect the cohomology ring, since we work with rational coefficients.

The semi-stable locus $\bar{\mathbb{V}}^{\rm ss}(T,\Sigma_{(d,\chi^l)})$ corresponds to the representation

$$\bar{\rho}: T \longrightarrow \mathrm{Gl}(\mathbb{C}\oplus V), \quad \bar{\rho}(g)=\begin{pmatrix} \chi(g)^{-1/d} & 0 \\ 0 & \chi(g)^{-1/d}\rho(g)\end{pmatrix}.$$

If $\eta_1,\dots,\eta_r$ are the characters of the T-action on $\mathbb{V}\cong\mathbb{A}^r_{\mathbb{C}}$ (together with their multiplicities), then T acts on $\mathbb{A}^1_{\mathbb{C}}\times\mathbb{A}^r_{\mathbb{C}}$ by the characters

$$\bar{\eta}_0:=\chi^{-1/d}, \bar{\eta}_1:=\chi^{-1/d}\eta_1,\dots,\bar{\eta}_r:=\chi^{-1/d}\eta_r.$$

The irreducible components of the T-unstable locus in $\bar{\mathbb{V}}$ are of the form $\mathbb{P}(F)$, where $F\hookrightarrow\mathbb{C}\oplus\mathbb{C}^r$ runs over the set of unstable coordinate planes. For such a plane F, we denote $z(F)\subset\{0,1,\dots,r\}$ its defining equations, that is

$$F=\{x_j=0 \mid j\in z(F)\}\subset\mathbb{C}\oplus\mathbb{C}^r.$$

Then [3], théorème 2.1, says that

$$\mathrm{Ker}\Big(H^*_T(\bar{\mathbb{V}})\to H^*_T\left(\bar{\mathbb{V}}^{\rm ss}(T,\Sigma_{(d,\chi^l)})\right)\Big)=\left\langle \prod_{j\in z(F)}(h+\bar{\eta}_j)\;\middle|\; \begin{array}{l} F\subset\mathbb{C}\oplus\mathbb{C}^r \text{ is} \\ \text{unstable plane}\end{array}\right\rangle,$$

where $h:=c_1\big(\mathscr{O}_{\bar{\mathbb{V}}}(1)\big)$.

<u>*Claim*</u> $\jmath^*_T(h+\bar{\eta}_j)=\begin{cases}0 & \text{if } 0\in z(F);\\ \eta_j & \text{if } 0\notin z(F).\end{cases}$

Proof. (of claim) We notice first that $h+\bar{\eta}_j=c_1^T\big(\mathscr{O}(1)_{\bar{\eta}_j}\big)$, where we have denoted $\mathscr{O}(1)_{\bar{\eta}_j}$ the T-linearized invertible sheaf $\mathscr{O}_{\bar{\mathbb{V}}}(1)$ endowed with the T-action through the character $\bar{\eta}_j$. More precisely, the T-action on the geometric line bundle is

$$T\times\mathcal{O}_{\bar{\mathbb{V}}}(1)\longrightarrow\mathcal{O}_{\bar{\mathbb{V}}}(1), \quad t\times[(a,v),z]:=[t\times(a,v),\bar{\eta}_j(t)z].$$

A consequence of the commutative diagram (5.1), this time for T-equivariant cohomologies, is that for computing the restriction $\jmath^*_T(h+\bar{\eta}_j)\in H^2_T(\mathbb{V})$, we must determine the character with which T operates on $\jmath^*_0\mathcal{O}(1)_{\bar{\eta}_j}$. An immediate computation shows that it is

$$\bar{\eta}_0^{-1}\bar{\eta}_j=\begin{cases}1 & \text{if } j=0;\\ \eta_j & \text{if } j\neq 0.\end{cases}$$

This finishes the proof of the claim. □

Back to the proof of the theorem. Since $\mathbb{V}^{\rm ss}(T,\chi)=\mathbb{V}\cap\bar{\mathbb{V}}^{\rm ss}\big(T,\Sigma_{(d,\chi^\ell)}\big)$, it follows that the intersections $\mathbb{V}\cap\mathbb{P}(F)$, with F unstable plane in $\mathbb{C}\oplus V$, coincide with the linear spaces $\mathbb{E}(\lambda)\hookrightarrow\mathbb{V}$, $\lambda\in\mathcal{F}(\chi)$. For any such F we distinguish two possibilities: either $0\in z(F)$ or $0\notin z(F)$. The claim above implies that

$$\jmath^*_T\Big(\prod_{j\in z(F)}(h+\bar{\eta}_j)\Big)=\begin{cases}0 & \text{if } 0\in z(F);\\ \displaystyle\prod_{j\in z(F)}\eta_j & \text{if } j\neq 0.\end{cases}$$

Now, on one hand we know that $\mathbb{V}\cap\mathbb{P}(F)$ coincides with some $\mathbb{E}(\lambda)$, $\lambda\in\mathcal{F}(\chi)$, and all the $\mathbb{E}(\lambda)$'s occur in this way, and on the other hand that the defining equations of $\mathbb{V}\cap\mathbb{P}(F)$ are $\{x_j=0, j\in z(F)\}$. These two facts imply that

$$\prod_{j\in z(F)}\eta_j=[\mathbb{E}(\lambda)]_T,\ \text{for appropriate } \lambda\in\mathcal{F}(\chi),$$

which finishes the proof of the theorem. □

6. Construction of families of quotients

Let us recall that to any scheme S and locally free sheaf $\mathscr{F}\to S$, one can associate the Grassmannian $\mathrm{Grass}(\mathscr{F},d)\to S$ of d-dimensional quotients of $\mathscr{E}$. With this motivation in mind, we wish to address the problem of constructing families of varieties, which are invariant quotients of affine spaces, over arbitrary bases. Our construction relies on Seshadri's construction of quotients for actions of reductive group schemes.

We fix a complex, connected, reductive group $G_{\mathbb{C}}$ and a complex representation $\rho: G_{\mathbb{C}}\to \mathrm{Gl}(V_{\mathbb{C}})$ of it with the property that $\mathbb{C}[\mathbb{V}_{\mathbb{C}}]^{G_{\mathbb{C}}}=\mathbb{C}$. Then $V_{\mathbb{C}}$ decomposes uniquely, as a $G_{\mathbb{C}}$-module, into its isotypical components

$$V_{\mathbb{C}}=\bigoplus_{\omega\in\mathcal{X}}\big(M^{\vee}_{\mathbb{C},\omega}\big)^{\oplus\nu_\omega},$$

where the $M^{\vee}_{\mathbb{C},\omega}$'s are simple $G_{\mathbb{C}}$-modules. The dual is taken here for simplifying the forthcomming notations. We denote by $\rho_\omega: G_{\mathbb{C}}\to\mathrm{Gl}(M_{\mathbb{C},\omega})$ the corresponding representations. Further, for each ω, we choose an *admissible lattice* $M_\omega\subset M_{\mathbb{C},\omega}$ (see [5], page 225). Chevalley associates to this data, in [5], section 4, a reductive group scheme $\mathbf{G}\to\operatorname{Spec}\mathbb{Z}$ which contains a maximal torus, and whose geometric fibres are connected, reductive groups having the same root system as $G_{\mathbb{C}}$. Moreover, the representations ρ_ω extend to

$$\rho_\omega:\mathbf{G}\longrightarrow\mathbf{Gl}(M^{\vee}_\omega):=\operatorname{Spec}\Big(\big(\operatorname{Sym}^{\bullet}_{\mathbb{Z}}\operatorname{End}_{\mathbb{Z}}(M_\omega)\big)[\det{}^{-1}]\Big).$$

We fix a character $\chi:\mathbf{G}\to\mathbf{G}_m=\operatorname{Spec}\mathbb{Z}[t,t^{-1}]$. Then for any field K, we get an induced character $\chi_K: G_K\to G_{m,K}=\operatorname{Spec}K[t,t^{-1}]$.

Lemma 6.1. *If $G\to\mathrm{Gl}(\nu;\mathbb{C})$ is a representation such that $\mathbb{C}[t_1,\dots,t_\nu]^{G_{\mathbb{C}}}=\mathbb{C}$, then for any reduced ring B holds $B[t_1,\dots,t_\nu]^{\mathbf{G}_B}=B$.*

Proof. Assume the contrary, that there is $f\in B[t_1,\dots,t_\nu]^{\mathbf{G}_B}$, homogeneous with $\deg f>0$. Since B is reduced, there is $\mathfrak{q}\in\operatorname{Spec}B$ such that the coefficients of f are not contained in $\mathfrak{q}$. We let K to be the algebraic closure of quotient field $Q(B/\mathfrak{q})$, and we observe that the polynomial F_K obtained by extending the coefficients of f is non-zero, G_K-invariant, $\deg f_K>0$. Using [25], theorem 1, we deduce that there is a homogeneous $F\in\mathbb{Z}[t_1,\dots,t_\nu]^{\mathbf{G}}$ with $\deg F>0$; this contradicts our hypothesis. □

We consider further a ring R, which is a finite algebra over a universally Japanese ring; the examples we have in mind are $R=\mathbb{Z}$ (for mixed characteristic methods) and $\mathbb{Z}\big[[t]\big]$ (for the study of deformations).

Let $S \to \operatorname{Spec} R$ be a separated and reduced scheme of finite type. For $\omega \in \Omega$, we consider a locally free sheaf $\mathscr{V}_\omega \to S$ with $\operatorname{rank}_{\mathscr{O}_X} \mathscr{V}_\omega = \nu_\omega$, and define

$$\mathscr{V} := \bigoplus_{\omega} M_\omega^\vee \otimes_{\mathbb{Z}} \mathscr{V}_\omega \to S.$$

We denote $\mathbf{V} := \underline{\operatorname{Spec}}\big(\operatorname{Sym}^\bullet_{\mathscr{O}_S} \mathscr{V}^\vee\big) \xrightarrow{\pi} S$ the corresponding geometric vector bundle.

Theorem 6.2. (i) *There is a natural action $\mathbf{G} \times_{\operatorname{Spec}\mathbb{Z}} \mathbf{V} \to \mathbf{V}$ of the group scheme $\mathbf{G} \to \operatorname{Spec}\mathbb{Z}$, which covers the trivial action on S.*
(ii) *There is a $\mathbf{G}$-invariant open subscheme $\mathbf{V}^{\mathrm{ss}}(\mathbf{G},\chi) \subset \mathbf{V}$ satisfying the following properties:*
(ii$_\mathrm{a}$) *There is a categorical quotient $(\mathbf{Y}, q)$ for the $\mathbf{G}$-action on $\mathbf{V}^{\mathrm{ss}}(\mathbf{G},\chi)$. Moreover, there is a natural isomorphism*

$$\mathbf{Y} \cong \underline{\operatorname{Proj}}\bigg(\bigoplus_{n\geqslant 0} (\pi_*\mathscr{O}_{\mathbf{V}})^{\mathbf{G}}_{\chi^n}\bigg) \longrightarrow S,$$

and $\mathbf{Y}$ is projective over S;
(ii$_\mathrm{b}$) *For any algebraically closed field K, and any morphism $\operatorname{Spec} K \to S$, $\operatorname{Spec} K \times_S \mathbf{V}^{\mathrm{ss}}(\mathbf{G},\chi)$ consists of the χ_K-semi-stable points of $\operatorname{Spec} K \times_S \mathbf{V}$;*

Proof. The proof will be divided in three steps: first we show that there is a natural $\mathbf{G}$-action on $\mathbf{V}$, next that the quotient exists locally, and finally we glue the local quotients together.

For proving that there is a global $\mathbf{G}$-action on $\mathbf{V}$, is enough to prove that $\mathbf{Gl}(M_\omega)$ acts, for each $\omega \in \Omega$. We observe that the ring homomorphism

$$\begin{array}{rcl} \operatorname{Sym}^\bullet_{\mathscr{O}_S}\big(M_\omega \otimes_{\mathbb{Z}} \mathscr{V}_\omega^\vee\big) & \longrightarrow & \operatorname{Sym}^\bullet_{\mathscr{O}_S}\big(\mathscr{O}_S \otimes_{\mathbb{Z}} \operatorname{End}_{\mathbb{Z}}(M_\omega)\big) \otimes_{\mathscr{O}_S} \operatorname{Sym}^\bullet_{\mathscr{O}_S}\big(M_\omega \otimes_{\mathbb{Z}} \mathscr{V}_\omega^\vee\big) \\ m \otimes v & \longmapsto & \mu_\omega(m) \otimes v, \end{array}$$

with $\mu_\omega : \operatorname{Sym}^\bullet_{\mathbb{Z}} M_\omega \to \operatorname{Sym}^\bullet_{\mathbb{Z}} \operatorname{End}_{\mathbb{Z}}(M_\omega) \otimes \operatorname{Sym}^\bullet_{\mathbb{Z}} M_\omega$ the co-multiplication, defines the $\mathbf{Gl}(M_\omega)$ action on $\underline{\operatorname{Spec}}(\operatorname{Sym}^\bullet_{\mathscr{O}_S} M_\omega \otimes_{\mathbb{Z}} \mathscr{V}_\omega^\vee)$. Since

$$\mathbf{V} = \times_{S,\omega\in\Omega} \underline{\operatorname{Spec}}(\operatorname{Sym}^\bullet_{\mathscr{O}_S} M_\omega \otimes_{\mathbb{Z}} \mathscr{V}_\omega^\vee),$$

$\mathbf{Gl}(M_\omega^\vee)$ acts on $\mathbf{V}$ too.

The theory developed in [25] works for closed subvarieties of affine spaces (see Remark 7(b) in *loc. cit.*). In our situation $\mathbf{V}$ can not be embedded as a $\mathbf{G}$-invariant closed subset of an affine space over $\operatorname{Spec} R$, unless each $\mathscr{V}_\omega \to S$ is globally generated. What we do instead, is to construct the quotients locally over S, and glue them together.

Let $(S_i = \operatorname{Spec} B_i)_i$ be a finite covering of S with open, affine subsets, which are moreover trivializing for all the $\mathscr{V}_\omega$'s. For every index i, we find there is an integer $n_i \geqslant 1$ such that $S_i \hookrightarrow \mathbb{A}^{n_i}_R$. It follows that

$$\begin{aligned} \mathbf{V}_i &:= \times_{S,\omega\in\Omega} \underline{\operatorname{Spec}}\big(\operatorname{Sym}^\bullet_{\mathscr{O}_S} M_\omega \otimes_{\mathbb{Z}} \mathscr{V}_\omega^\vee\big)|_{S_i} \\ &\hookrightarrow \times_{\mathbb{A}^{n_i}_R,\omega\in\Omega} \underline{\operatorname{Spec}}\big(\operatorname{Sym}^\bullet_{\mathscr{O}_{\mathbb{A}^{n_i}_R}} M_\omega \otimes_{\mathbb{Z}} \mathscr{O}^{\oplus\nu_\omega}_{\mathbb{A}^{n_i}_R}\big) = \mathbb{A}^{n_i+\nu}_R, \end{aligned}$$

with $\nu := \sum_\omega \operatorname{rank} M_\omega \cdot \nu_\omega$, and moreover this embedding is $\mathbf{G}$-equivariant.

For $R_i := \Gamma\big(\mathbb{A}^1_S \times_S \mathbf{V}_i, \mathscr{O}_{\mathbb{A}^1_S \times_S \mathbf{V}_i}\big) \cong B_i[t, t_1, \ldots, t_\nu]$, [25], theorem 2, implies that

$$R_i^{\mathbf{G}} = \bigoplus_{m\geqslant 0} t^m \cdot B_i[t_1, \ldots, t_\nu]^{\mathbf{G}}_{\chi^m} \subset R_i$$

is a finitely generated B_i-algebra. We define the χ-unstable locus $\mathbf{V}_i^{\mathrm{us}}(\mathbf{G},\chi)\hookrightarrow\mathbf{V}_i$ to be the closed subscheme defined by the ideal

$$\big\langle B_i[t_1,\dots,t_\nu]^{\mathbf{G}}_{\chi^m}\,;\,m\geqslant 1\big\rangle\subset B_i[t_1,\dots,t_\nu]$$

and let $\mathbf{V}_i^{\mathrm{ss}}(\mathbf{G},\chi)$ to be its complement. The gluing argument of [22], theorem 1.10, together with [25], theorem 3 and remark 8, imply that the categorical quotient $\mathbf{Y}_i$ of $\mathbf{V}_i^{\mathrm{ss}}(\mathbf{G},\chi)$ by $\mathbf{G}$ exists, and it is isomorphic to $\underline{\mathrm{Proj}}\big(\bigoplus_{m\geqslant 0}B_i[t_1,\dots,t_\nu]^{\mathbf{G}}_{\chi^m}\big)$. Since $B_i[t_1,\dots,t_\nu]^{\mathbf{G}}=B_i$, $\mathbf{Y}_i$ is projective over $\mathrm{Spec}(B_i)$.

The flat base change property [25], lemma 2, implies that for two open subsets $S_i,S_j\subset S$ holds $\mathbf{V}_i^{\mathrm{ss}}(\mathbf{G},\chi)|_{S_i\cap S_j}=\mathbf{V}_j^{\mathrm{ss}}(\mathbf{G},\chi)|_{S_i\cap S_j}$. It follows that there is a well-defined open subset $\mathbf{V}^{\mathrm{ss}}(\mathbf{G},\chi)\subset\mathbf{V}$ of χ-semi-stable points. Finally, using the universality property of categorical quotients, we glue together the $\mathbf{V}_i^{\mathrm{ss}}(\mathbf{G},\chi)\to\mathbf{Y}_i$'s into a scheme $\mathbf{Y}$, and this comes with a natural morphism $\mathbf{V}^{\mathrm{ss}}(\mathbf{G},\chi)\to\mathbf{Y}$.

It remains to prove the behaviour of the semi-stable locus under base change; the question being local on S, we may assume that $S=\mathrm{Spec}\,B$. The inclusion $K\times_B\mathbf{V}^{\mathrm{ss}}(\mathbf{G},\chi)\subset\mathbf{V}_K^{\mathrm{ss}}(G_K,\chi_K)$ is trivial, so that we prove the converse. Consider a K-valued point $x_0\in\mathbf{V}_K^{\mathrm{ss}}(G_K,\chi_K)$, and a homogeneous $f\in K[t_1,\dots,t_\nu]^{G_K}_{\chi_K^m}$, $m\geqslant 1$, such that $f(x_0)\neq 0$. Then $F_K:=t^{\deg f}f\in K[t,t_1,\dots,t_\nu]^{G_K}$, and applying [25], theorem 1, we find a homogeneous $F\in B[t,t_1,\dots,t_\nu]^{\mathbf{G}_B}$ such that $\deg F>0$ and $F(1,x_0)\neq 0$. By lemma 6.1, $B[t_1,\dots,t_\nu]^{\mathbf{G}_B}=B$, so that $F\in\bigoplus_{m\geqslant 1}t^mB[t_1,\dots,t_\nu]^{\mathbf{G}_B}_{\chi^m}$, and therefore $x_0\in K\times_B\mathbf{V}^{\mathrm{ss}}(G,\chi)$. □

One of the most common situation where the construction applies is that of the Grassmann bundle of d-dimensional quotients of a locally free sheaf $\mathscr{F}\to S$. In this case $\mathscr{V}:=\mathrm{Hom}_{\mathbb{Z}}(\mathscr{F},\mathbb{Z}^d)\cong\mathscr{F}^\vee\otimes_{\mathbb{Z}}\mathbb{Z}^d$, $\mathbf{G}:=\mathbf{Gl}(d)$, and $\chi:=\det$.

As a first application, we describe how varies the cone decomposition obtained in section 3 with the characteristic of the ground field.

Corollary 6.3. *Assume $S=\mathrm{Spec}\,\mathbb{Z}$, $\mathbf{V}=\mathrm{Spec}\big(\mathrm{Sym}^\bullet_{\mathbb{Z}}M\big)$, and $\chi,\varepsilon:\mathbf{G}\to\mathbf{G}_m$ are characters such that $\varepsilon_{\mathbb{C}}\in C(\chi_{\mathbb{C}})$. Then there is a prime $p_0>0$ depending on ε, such that $\varepsilon_K\in C(\chi_K)$ for any algebraically closed field K of characteristic zero or $\mathrm{char}\,K>p_0$. In particular, if $C(\varepsilon_{\mathbb{C}})=C(\chi_{\mathbb{C}})$ then the same holds over fields of large characteristic.*

Proof. The first remark is that $\mathbf{V}_{\mathbb{Q}}^{\mathrm{ss}}(G_{\mathbb{Q}},\chi_{\mathbb{Q}})\subset\mathbf{V}_{\mathbb{Q}}^{\mathrm{ss}}(G_{\mathbb{Q}},\varepsilon_{\mathbb{Q}})$: indeed, otherwise there would exist $\mathfrak{q}\in\mathrm{Spec}\big(\mathrm{Sym}^\bullet_{\mathbb{Q}}M_{\mathbb{Q}}\big)$ such that

$$\mathfrak{q}\in\mathbf{V}_{\mathbb{Q}}^{\mathrm{ss}}(G_{\mathbb{Q}},\chi_{\mathbb{Q}})\quad\text{and}\quad\mathfrak{q}\supset\big\langle(\mathrm{Sym}^\bullet_{\mathbb{Q}}M_{\mathbb{Q}})^{G_{\mathbb{Q}}}_{\varepsilon_{\mathbb{Q}}^m}\,;\,m\geqslant 1\big\rangle.$$

Using theorem 6.2 (ii$_{\mathrm{b}}$), we obtain that $\mathfrak{q}_{\mathbb{C}}\in\mathbf{V}_{\mathbb{C}}^{\mathrm{ss}}(G_{\mathbb{C}},\chi_{\mathbb{C}})$ and

$$\mathfrak{q}_{\mathbb{C}}\supset\sqrt{\big\langle(\mathrm{Sym}^\bullet_{\mathbb{Q}}M_{\mathbb{Q}})^{G_{\mathbb{Q}}}_{\varepsilon_{\mathbb{Q}}^m}\,;\,m\geqslant 1\big\rangle_{\mathbb{C}}}=\sqrt{\big\langle(\mathrm{Sym}^\bullet_{\mathbb{C}}M_{\mathbb{C}})^{G_{\mathbb{C}}}_{\varepsilon_{\mathbb{C}}^m}\,;\,m\geqslant 1\big\rangle},$$

which contradicts our hypothesis.

Applying theorem 6.2 (ii$_{\mathrm{b}}$) again, we deduce the validity of the corollary in characteristic zero. To settle the problem in positive characteristic, let us notice that the irreducible components of $\mathbf{V}^{\mathrm{us}}(\mathbf{G},\varepsilon)_{\mathrm{red}}\hookrightarrow\mathbf{V}$ are divided in two disjoint groups: those whose generic points project onto $\langle 0\rangle\in\mathrm{Spec}\,\mathbb{Z}$, and those which do not (*i.e.* the residue field has characteristic zero or strictly positive). Let $U\subset\mathrm{Spec}\,\mathbb{Z}$ be the complement of the finite set of primes where we have such 'bad reduction'. Since $\mathbf{V}_{\mathbb{Q}}^{\mathrm{us}}(G_{\mathbb{Q}},\varepsilon_{\mathbb{Q}})\hookrightarrow\mathbf{V}_{\mathbb{Q}}^{\mathrm{us}}(G_{\mathbb{Q}},\chi_{\mathbb{Q}})$, we deduce that $\mathbf{V}^{\mathrm{us}}(\mathbf{G},\varepsilon)|_U\hookrightarrow\mathbf{V}^{\mathrm{us}}(\mathbf{G},\chi)|_U$. □

We specialize now to the case where S is a reduced and irreducible K-scheme of finite type, where K is an algebraically closed field. Our construction yields in this case a locally trivial fibration $\mathbf{Y} \to S$, whose fibres are isomorphic to $\mathbb{V}/\!\!/_{\chi}G$, and we interested in computing its Chow ring. Reasoning locally over S, we see that the $\chi|_T$-unstable locus of $\mathbf{V}$ for the induced T-action is the union

$$\mathbf{V}^{\mathrm{us}}(T,\chi) = \bigcup_{\lambda\in\mathcal{F}(\chi)} \mathbf{E}(\lambda), \quad \mathbf{E}(\lambda) := \underline{\mathrm{Spec}}\big(\mathscr{E}(\lambda)^{\vee}\big) \hookrightarrow \mathbf{V},$$

where $\mathscr{E}(\lambda) \subset \mathscr{V}$ are locally free subsheaves. The χ-unstable locus for the G-action is then

$$\mathbf{V}^{\mathrm{us}}(G,\chi) = \bigcup_{\lambda\in\mathcal{F}(\chi)} G\cdot\mathbf{E}(\lambda).$$

We still denote by $\wp_G : A_*(S)_{\mathbb{Q}} \otimes_{\mathbb{Q}} (A^T_*)_{\mathbb{Q}} \to A_*(S)_{\mathbb{Q}} \otimes_{\mathbb{Q}} (A^T_*)^W_{\mathbb{Q}}$ the homomorphism obtained from $\wp_G$, as defined in section 4, by extending the scalars.

Theorem 6.4. *Consider $\chi \in \mathcal{X}^*(G)$ such that $\mathbb{V}^{\mathrm{ss}}(G,\chi) = \mathbb{V}^{\mathrm{s}}_{(0)}(G,\chi)$. Then holds:*
(i) *the quotient $\mathbf{Y} := \mathbf{V}^{\mathrm{ss}}(G,\chi)/G$ is geometric;*
(ii) *the Chow ring*

$$A_*(\mathbf{Y})_{\mathbb{Q}} \cong A_*(S)_{\mathbb{Q}} \otimes_{\mathbb{Q}} (A^T_*)^W_{\mathbb{Q}} \Big/ \wp_G\big\langle \mathsf{e}^T(\mathscr{V}/\mathscr{E}(\lambda))\,;\, \lambda\in\mathcal{F}(\chi)\big\rangle_{\mathbb{Q}}.$$

(iii) *Assume moreover that $K = \mathbb{C}$. Then*

$$H^*(\mathbf{Y};\mathbb{Q}) \cong H^*(S)_{\mathbb{Q}} \otimes_{\mathbb{Q}} (H^*_T)^W_{\mathbb{Q}} \Big/ \wp_G\big\langle \mathsf{e}^T(\mathscr{V}/\mathscr{E}(\lambda))\,;\, \lambda\in\mathcal{F}(\chi)\big\rangle_{\mathbb{Q}}.$$

Proof. The quotient $\mathbf{V}^{\mathrm{ss}}(G,\chi) \to \mathbf{Y}$ is geometric because it is so locally over S. Now we prove the second statement: since G acts with finite stabilizers, we have the exact sequence

$$A^G_*\Bigg(\bigcup_{\lambda\in\mathcal{F}(\chi)} G\cdot\mathbf{E}(\lambda)\Bigg)_{\mathbb{Q}} \longrightarrow A^G_*(\mathbf{V})_{\mathbb{Q}} \longrightarrow A^G_*\big(\mathbf{V}^{\mathrm{ss}}(G,\chi)\big)_{\mathbb{Q}} \cong A_{*-\dim G}(\mathbf{Y})_{\mathbb{Q}} \longrightarrow 0.$$

Repeating the proof of proposition 4.4 in our relative setting, and using the excess intersection formula [11], example 6.3.5, we find that the image of the homomorphism $A^G_*\big(\mathbf{E}(\lambda)\big)_{\mathbb{Q}} \to A^G_*(\mathbf{V})_{\mathbb{Q}}$ is the ideal $\wp_G\big\langle \mathsf{e}^T\big(\mathscr{V}/\mathscr{E}(\lambda)\big)\big\rangle_{\mathbb{Q}}$. This proves the second claim.

The proof of the third statement is divided in three steps:

– We recall that $\mathbf{Y} \to S$ is a locally trivial fibre bundle, with fibre $Y := \mathbb{V}/\!\!/_{\chi}G$. According to proposition 4.6, the Chow ring of Y, hence its cohomology ring too, is generated by characteristic classes of (virtual) representations of G. In the relative setting, one may still consider the locally free sheaves associated to representations of G, and the corresponding characteristic classes yield a cohomology extension of the fibre. We conclude that the Leray-Hirsch theorem applies to $\mathbf{Y} \to S$, and that the long exact sequence for the cohomology of the pair $\big(\mathbf{V}, \mathbf{V}^{\mathrm{ss}}(\mathbb{V},\chi)\big)$ splits into short exact sequences

$$0 \longrightarrow H^*_G\big(\mathbf{V}, \mathbf{V}^{\mathrm{ss}}(G,\chi);\mathbb{Q}\big) \xrightarrow{\imath^*_S} H^*_G(\mathbf{V};\mathbb{Q}) \xrightarrow{\jmath^*_S} H^*_G\big(\mathbf{V}^{\mathrm{ss}}(G,\chi);\mathbb{Q}\big) \longrightarrow 0.$$

– Fibrewise holds: the ring homomorphism $H^*_G(\mathbb{V};\mathbb{Q}) \to H^*_G\big(\mathbb{V}^{\mathrm{ss}}(G,\chi);\mathbb{Q}\big) \cong H^*(Y)$ is surjective, the short sequence

$$0 \longrightarrow H^*_G\big(\mathbb{V}, \mathbb{V}^{\mathrm{ss}}(G,\chi);\mathbb{Q}\big) \xrightarrow{\imath^*} H^*_G(\mathbb{V};\mathbb{Q}) \xrightarrow{\jmath^*} H^*_G\big(\mathbb{V}^{\mathrm{ss}}(G,\chi);\mathbb{Q}\big) \longrightarrow 0,$$

is exact, and moreover $\imath^* H^*_G\big(\mathbb{V},\mathbb{V}^{\rm ss}(G,\chi);\mathbb{Q}\big)=\wp_G\big\langle[\mathbb{E}(\lambda)]_T\,;\,\lambda\in\mathcal{F}(\chi)\big\rangle_{\mathbb{Q}}$.

– The restriction to the fibres of the fibre bundle pair $\big(\mathbf{V},\mathbf{V}^{\rm ss}(G,\chi)\big)\overset{\pi}{\longrightarrow}S$ of the ideal $\wp_G\big\langle \mathsf{e}^T\big(\mathscr{V}/\mathscr{E}(\lambda)\big)\big\rangle_{\mathbb{Q}}\subset H^*_G(\mathbf{V};\mathbb{Q})$, $\lambda\in\mathcal{F}(\chi)$, equals $\wp_G\big\langle[\mathbb{E}(\lambda)]_T\big\rangle$. Moreover, we remark that the restriction of $\pi^*\big(\mathscr{V}/\mathscr{E}(\lambda)\big)$ to $\mathbf{V}^{\rm ss}(G,\chi)$ has a nowhere vanishing, T-equivariant section: it is the composition of the canonical section $\mathbf{V}\to\pi^*\mathscr{V}$ with the projection $\mathscr{V}\to\mathscr{V}/\mathscr{E}(\lambda)$.

It follows that $\jmath_S^*\mathsf{e}^T\big(\mathscr{V}/\mathscr{E}(\lambda)\big)=0$, and therefore $\jmath_S^*\big[\wp_G\big\langle\mathsf{e}^T\big(\mathscr{V}/\mathscr{E}(\lambda)\big)\big\rangle_{\mathbb{Q}}\big]=0$. Applying the Leray-Hirsch theorem to the pair $\big(\mathbf{V},\mathbf{V}^{\rm ss}(\mathbb{V},\chi)\big)\to S$, we deduce:

$$\imath_S^*\big[H^*_G\big(\mathbf{V},\mathbf{V}^{\rm ss}(G,\chi);\mathbb{Q}\big)\big]=\wp_G\big\langle\mathsf{e}^T\big(\mathscr{V}/\mathscr{E}(\lambda)\big)\,;\,\lambda\in\mathcal{F}(\chi)\big\rangle_{\mathbb{Q}}.$$

Since $\jmath_S^*$ is a ring homomorphisms, our claim follows. □

7. The Picard group, the ample cone and the cohomology of line bundles

In this section we compute first the Picard group and the ample cone of the quotient varieties that we obtain. Secondly we prove the vanishing of the higher cohomology groups for nef line bundles.

Definition 7.1. Let $\chi\in\mathcal{X}^*(G)$ be a character, and consider the categorical quotient $q:\mathbb{V}^{\rm ss}(G,\chi)\to Y_\chi:=\mathbb{V}^{\rm ss}(G,\chi)/G$. For a further character $\varepsilon\in\mathcal{X}^*(G)$, we define

$$\mathscr{L}_{Y_\chi,\varepsilon}:=\big(q_*\mathscr{O}_{\mathbb{V}^{\rm ss}(G,\chi)}\big)^G_\varepsilon,\ \text{that is}\ \Gamma(U,\mathscr{L}_\varepsilon)=\Gamma\big(q^{-1}U,\mathscr{O}_{\mathbb{V}^{\rm ss}(G,\chi)}\big)^G_\varepsilon,\ \forall U\subset Y_\chi\ \text{open}.$$

Clearly $\mathscr{L}_{Y_\chi,\varepsilon}$ is torsion-free and coherent. If $\mathbb{V}^{\rm ss}(G,\chi)=\mathbb{V}^{\rm s}_{(0)}(G,\chi)$, $\mathscr{L}_{Y_\chi,\varepsilon}$ has rank one, and if moreover the G-action on $\mathbb{V}^{\rm ss}(G,\chi)$ is free, then $\mathscr{L}_{Y_\chi,\varepsilon}$ is actually invertible.

We recall from section 3 that for $\chi\in\mathcal{X}^*(G)$, $C(\chi)\in\Delta^G(\mathbb{V})$ denotes the cone containing χ in its relative interior.

For a character $\chi\in\mathcal{X}^*(G)$ such that $\mathbb{V}^{\rm ss}(G,\chi)=\mathbb{V}^{\rm s}_{(0)}(G,\chi)\neq\emptyset$, we define

$$\begin{aligned}\mathcal{F}'(\chi):=&\ \ \{\lambda\in\mathcal{F}(\chi)\mid \operatorname{codim}_{\mathbb{V}}\,G\cdot\mathbb{E}(\lambda)=1\}\\ =&\ \ \{\lambda\in\mathcal{F}(\chi)\mid \operatorname{codim}_{\mathbb{V}}\mathbb{E}(\lambda)+\operatorname{Stab}_G\mathbb{E}(\lambda)=\dim G+1\}.\end{aligned}$$

Denoting L_λ the Levi component of $\operatorname{Stab}_G\mathbb{E}(\lambda)$, we observe that for $\lambda\in\mathcal{F}'(\chi)$, $\deg[\mathbb{E}(\lambda)]_T-\deg\Delta_G/\Delta_{L_\lambda}=1$ (see section 4 for the notations). Since in this case $\wp_G\langle[\mathbb{E}(\lambda)]_T\rangle$ is a principal ideal, formula (4.2) implies that $\wp_G\langle[\mathbb{E}(\lambda)]_T\rangle_{\mathbb{Q}}=\langle\wp_G[\mathbb{E}(\lambda)]_T\rangle_{\mathbb{Q}}$. For shorthand, we will denote $\varepsilon(\lambda):=\wp_G[\mathbb{E}(\lambda)]_T\in(A^1_T)^W=\mathcal{X}^*(T)^W\cong\mathcal{X}^*(G)$.

Proposition 7.2. *Let $\chi\in\mathcal{X}^*(G)$ be a character such that $\mathbb{V}^{\rm ss}(G,\chi)=\mathbb{V}^{\rm s}_{(0)}(G,\chi)$. Then:*
(i) $\operatorname{Pic}(Y_\chi)_{\mathbb{Q}}\cong\mathcal{X}^*(G)_{\mathbb{Q}}/\big\langle\varepsilon(\lambda)\,;\,\lambda\in\mathcal{F}'(\chi)\big\rangle_{\mathbb{Q}}$.
In particular, if $\operatorname{codim}_{\mathbb{V}}\mathbb{V}^{\rm us}(G,\chi)\geqslant 2$*, then* $\operatorname{Pic}(Y_\chi)_{\mathbb{Q}}\cong\mathcal{X}^*(G)_{\mathbb{Q}}$.
(ii) *The ample cone of Y_χ is*

$$\operatorname{Pic}(Y_\chi)^{\rm ample}_{\mathbb{Q}}\cong\operatorname{int.}\hat{C}(\chi)_{\mathbb{Q}},$$

where $\hat{C}(\chi)_{\mathbb{Q}}:=\operatorname{Image}\big[\mathcal{X}^*(G)_{\mathbb{Q}}\cap C(\chi)\to\mathcal{X}^*(G)_{\mathbb{Q}}/\big\langle\varepsilon(\lambda)\,;\,\lambda\in\mathcal{F}'(\chi)\big\rangle_{\mathbb{Q}}\big]$.

Proof. (i) Indeed,

$$\begin{aligned}\operatorname{Pic}(Y_\chi)_{\mathbb{Q}}&=A^1(Y_\chi)_{\mathbb{Q}}=(A^1_T)^W_{\mathbb{Q}}\Big/(A^1_T)^W_{\mathbb{Q}}\cap\wp_G\big\langle[\mathbb{E}(\lambda)]_T\,;\,\lambda\in\mathcal{F}(\chi)\big\rangle_{\mathbb{Q}}\\&=\mathcal{X}^*(G)_{\mathbb{Q}}/\big\langle\varepsilon(\lambda)\,;\,\lambda\in\mathcal{F}'(\chi)\big\rangle_{\mathbb{Q}}.\end{aligned}$$

If and $\operatorname{codim}_{\mathbb{V}}\mathbb{V}^{\rm us}(G,\chi)\geqslant 2$, there is a short exact sequence

$$0\longrightarrow \operatorname{Pic}(Y_\chi)\xrightarrow{q^*}\mathcal{X}^*(G)\longrightarrow \mathcal{T}\longrightarrow 0,$$

with $\mathcal{T}$ a torsion module. Indeed, the pull-back by q of an invertible sheaf on Y_χ gives rise to a G-linearized invertible sheaf on $\mathbb{V}^{\rm ss}(G,\chi)$, and the codimension condition ensures that it uniquely extends to $\mathbb{V}$.
(ii) A character $\varepsilon\in\mathcal{X}^*(G)_{\mathbb{Q}}\cap\operatorname{int.}C(\chi)$ defines the same invariant quotient as χ, that is Y, and for $n\geqslant 1$ sufficiently large the sheaf $\mathscr{L}_{\varepsilon^n}\to Y$ is invertible and ample.

For $\lambda\in\mathcal{F}'(\chi)$, let $f_\lambda\in K[\mathbb{V}]$ be the equation of $G\cdot\mathbb{E}(\lambda)\hookrightarrow\mathbb{V}$; G acts on $\mathbb{Q}f_\lambda$ by a character, which is just the class $[G\cdot\mathbb{E}(\lambda)]_G\in A^*_G$, that is $\varepsilon(\lambda)$. We conclude that $f_\lambda\in K[\mathbb{V}]^G_{\varepsilon(\lambda)}$. We deduce further that for $\varepsilon'=\sum_{\lambda\in\mathcal{F}'(\chi)}k_\lambda\varepsilon(\lambda)$, with $k_\lambda\geqslant 0$,

$$\prod_{\lambda\in\mathcal{F}'(\chi)}f_\lambda^{k_\lambda}:\mathscr{L}_\varepsilon=\big(q_*\mathscr{O}_{\mathbb{V}^{\rm ss}(G,\chi)}\big)^{G,\varepsilon}\longrightarrow\big(q_*\mathscr{O}_{\mathbb{V}^{\rm ss}(G,\chi)}\big)^{G,\varepsilon\varepsilon'}=\mathscr{L}_{\varepsilon\varepsilon'}$$

is an isomorphism. This proves the inclusion of the right hand- into the left hand side.

For the converse inclusion, consider an ample line bundle $\mathscr{L}\to Y$. The first part implies that $\mathscr{L}\cong\mathscr{L}_\varepsilon$ for some $\varepsilon\in\mathcal{X}^*(G)$. Choose $n\geqslant 1$ such that the non-vanishing loci of the sections in $\mathscr{L}^{\otimes n}$ cover Y with affine open sets; for any $x\in\mathbb{V}^{\rm ss}(G,\chi)$, there is $f\in K[\mathbb{V}^{\rm ss}(G,\chi)]^G_{\varepsilon^n}$ such that $f(x)\neq 0$. We choose finitely many f's such that the non-vanishing loci cover $\mathbb{V}^{\rm ss}(G,\chi)$. The components of codimension one of $\mathbb{V}\setminus\mathbb{V}^{\rm ss}(G,\chi)$ are the $G\cdot\mathbb{E}(\lambda)$'s, with $\lambda\in\mathcal{F}'(\chi)$. Therefore we find a monomial $\phi:=\prod_{\lambda\in\mathcal{F}'(\chi)}f_\lambda^{k_\lambda}$ with the property that the ϕf's, with f as above, extend to regular functions on $\mathbb{V}$; they all belong to $K[\mathbb{V}]^G_{\varepsilon^n\cdot\prod_\lambda\varepsilon_\lambda^{k_\lambda}}$, and their non-vanishing locus contain $\mathbb{V}^{\rm ss}(G,\chi)$. This means that $\mathbb{V}^{\rm ss}(G,\chi)\subset\mathbb{V}^{\rm ss}(G,\varepsilon^n\cdot\prod_\lambda\varepsilon_\lambda^{k_\lambda})$, and therefore $\varepsilon^n\cdot\prod_\lambda\varepsilon_\lambda^{k_\lambda}\in C(\chi)$.

All together, proves that

$$\operatorname{int.}\hat{C}(\chi)_{\mathbb{Q}}\subset\operatorname{Pic}(Y_\chi)^{\rm ample}_{\mathbb{Q}}\subset\hat{C}(\chi)_{\mathbb{Q}}\subset\operatorname{Pic}(Y_\chi)_{\mathbb{Q}}.$$

The conclusion follows since the ample cone is open in the Picard group. □

Example 7.3. Consider a quiver $Q=(Q_0,Q_1,h,t)$, and let $V:=\bigoplus_{a\in Q_1}\operatorname{Hom}\big(U_{h(a)},U_{t(a)}\big)$ be a representation space for Q with dimension vector $(\alpha_q)_{q\in Q_0}$ (compare with example 3.11). For $S\subset Q_0$ we let $G_S:=\prod_{q\in S}\operatorname{Gl}(U_q)$, and consider the (partial) quiver representation $\rho_S:G_S\to\operatorname{Gl}(V)$.

Let $\chi\in\mathcal{X}^*(G_S)$ be a character. If χ does not lay in a wall for the G_S-action, and the codimension of $\mathbb{V}(G_S,\chi)\hookrightarrow\mathbb{V}$ is at least two, then the Picard group of $\mathbb{V}/\!\!/_\chi G_S$ is isomorphic to $\mathbb{Q}^S$. Let $C_{\min}(\chi)$ be the smallest cone whose boundaries are defined by the equations (3.3), and which contains χ in its interior. Then the ample cone of $V/\!\!/_\chi G$ *contains* $\operatorname{int.}C_{\min}(\chi)$ (but it may happen to be larger).

We are going to prove that the higher cohomology groups of nef line bundles vanish. We must say from the very beginning that our result is a more or less straightforward consequence of the Hochster-Roberts theorem.

For two characters $\chi, \varepsilon \in \mathcal{X}^*(G)$, with $\varepsilon \in C(\chi)$, the inclusion $\mathbb{V}^{\rm ss}(G,\chi) \subset \mathbb{V}^{\rm ss}(G,\varepsilon)$ induces a morphism on the quotient level

$$\begin{array}{ccc} \mathbb{V}^{\rm ss}(G,\chi) & \longrightarrow & \mathbb{V}^{\rm ss}(G,\varepsilon) \\ {\scriptstyle q_\chi}\downarrow & & \downarrow{\scriptstyle q_\varepsilon} \\ Y_\chi & \xrightarrow{\ \psi\ } & Y_\varepsilon \end{array} \tag{7.1}$$

which is projective and open. In particular, ψ is surjective.

Lemma 7.4. (i) *For $f \in K[\mathbb{V}]^G_{\varepsilon^m}$, with $m \geqslant 1$, the pre-image under ψ of the affine, open subset $D(f, Y_\varepsilon)/G \subset Y_\varepsilon$ is isomorphic to* $\mathrm{Proj}\big(\underset{n\geqslant 0}{\oplus}(f^{-1}K[\mathbb{V}])^G_{\chi^n}\big)$.
(ii) *Assume* $\mathrm{char}\, K = 0$. *Then* $\mathrm{R}^i\psi_*\mathscr{O}_{Y_\chi} = 0, \quad \forall\, i > 0$.

Proof. (i) Is a direct consequence of the definition.
(ii) Is enough to prove that $\big(\mathrm{R}^i\psi_*\mathscr{O}_{Y_\chi}\big)\big(D(f,Y_\varepsilon)\big) = H^i\big(\psi^{-1}D(f,Y_\varepsilon), \mathscr{O}_{Y_\chi}\big) = 0$, for any $f \in K[\mathbb{V}]^G_{\varepsilon^m}$, $m \geqslant 1$. We have just seen that $\psi^{-1}D(f,Y_\varepsilon) = \mathrm{Proj}\, R$, with

$$R = \oplus_{n\geqslant 0} R_n := \bigoplus_{n\geqslant 0}(f^{-1}K[\mathbb{V}])^G_{\chi^n} \text{ and also } \bigoplus_{n\geqslant 0}(f^{-1}K[\mathbb{V}])^G_{\chi^n} = K[\mathbb{A}^1_K \times D(f)]^G.$$

Since G is linearly reductive, and $\mathbb{A}^1_K \times D(f)$ is a smooth, affine variety, we may apply [19], theorem 3.4: the positively graded part of the local cohomology groups $\big[H^i_{R_+}(R)\big]_{\geqslant 0}$ vanish, for all $i \geqslant 0$, where $R_+ := \oplus_{n\geqslant 1} R_n$. The Grothendieck-Serre correspondence [4], theorem 20.4.4, says that

$$\Big[H^{i+1}_{R_+}(R)\Big]_{\geqslant 0} \cong \bigoplus_{j\geqslant 0} H^i\big(\mathrm{Proj}\, R, \mathscr{O}_{\mathrm{Proj}\, R}(j)\big), \quad \forall\, i \geqslant 1,$$

and the vanishing of the degree zero part implies $H^i\big(\psi^{-1}D(f,Y_\varepsilon), \mathscr{O}_{Y_\chi}\big) = 0$. □

Lemma 7.5. *Let $\chi \in \mathcal{X}^*(G)$ such that* $\mathrm{codim}_{\mathbb{V}}\mathbb{V}^{\rm us}(G,\chi) \geqslant 2$, *and let $\varepsilon \in \mathcal{X}^*(G) \cap C(\chi)$. Then holds:*
(i) $\psi_*\mathscr{L}_{Y_\chi,\varepsilon} = \mathscr{L}_{Y_\varepsilon,\varepsilon}$;
(ii) *If $\mathscr{L}_{Y_\varepsilon,\varepsilon}$ is invertible, the natural homomorphism $\psi^*\mathscr{L}_{Y_\varepsilon,\varepsilon} = \psi^*\psi_*\mathscr{L}_{Y_\chi,\varepsilon} \to \mathscr{L}_{Y_\chi,\varepsilon}$ is an isomorphism.*

Proof. One simply applies the definition. □

We fix $\mathbf{G} \to \mathrm{Spec}\, \mathbb{Z}$ a reductive group scheme, which contains a maximal torus, and whose geometric fibres are connected, reductive groups, all having the same root system; we consider a representation $\mathbf{G} \to \mathbf{Gl}(M^\vee)$, as constructed at the beginning of the previous section, and assume that $\big(\mathrm{Sym}^\bullet_{\mathbb{Z}} M\big)^{\mathbf{G}} = \mathbb{Z}$. We define $\mathbf{V} := \mathrm{Spec}\big(\mathrm{Sym}^\bullet_{\mathbb{Z}} M\big)$, and, for a field K, we let $\mathbb{V}_K := \mathbf{V} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\, K = \mathrm{Spec}\big(\mathrm{Sym}^\bullet_K M \otimes_{\mathbb{Z}} K\big)$. Further, we fix a character $\chi : \mathbf{G} \to \mathbf{G}_m$ such that $\mathrm{codim}_{\mathbb{V}_{\mathbb{C}}}\mathbb{V}^{\rm us}_{\mathbb{C}}(G_{\mathbb{C}}, \chi_{\mathbb{C}}) \geqslant 2$, and we denote $Y_{\chi,K} := \mathbb{V}_K /\!\!/_{\chi_K} G_K = \mathrm{Proj}\, K[\mathbb{V}_K]^{G_K,\chi_K}$.

Finally, we consider a character $\varepsilon : \mathbf{G} \to \mathbf{G}_m$ such that $\varepsilon_{\mathbb{C}} \in C(\chi_{\mathbb{C}})$. Corollary 6.3 implies that there is an open subset $U \subset \mathrm{Spec}\,\mathbb{Z}$ such that $\mathbf{V}^{\rm ss}(\mathbf{G},\chi)|_U \subset \mathbf{V}^{\rm ss}(\mathbf{G},\varepsilon)|_U$, and consequently we obtain a commutative diagram similar to (7.1), defined over U.

Lemma 7.6. *For any field K, there is a natural isomorphism*

$$\big(\mathrm{Sym}^\bullet_{\mathbb{Z}}(\mathbb{Z} \oplus M)\big)^{\mathbf{G},\chi} \otimes_{\mathbb{Z}} K \xrightarrow{\ \cong\ } \big(\mathrm{Sym}^\bullet_K(K \oplus M_K)\big)^{G_K,\chi_K}.$$

In particular, there is $c>0$ such that $\mathscr{L}_{Y_\chi,\chi_K^c}\to Y_{\chi,K}$ is invertible for all fields K.

Proof. The second statement is indeed a consequence of the first: choose $c>0$ such that $\big(\mathrm{Sym}^\bullet_{\mathbb{Z}}(\mathbb{Z}\oplus M)\big)^{\mathbf{G},\chi^c}$ is generated in degree one, and apply base change.

The proof of the first part is inspired from [25], lemma 2. We observe that *loc. cit.* implies that is enough to prove our statement for $\mathbb{F}_p:=\mathbb{Z}/\langle p\rangle$, with $p>0$ prime.

The choice of a $\mathbb{Z}$-basis in M induces the isomorphism $\mathrm{Sym}^\bullet_{\mathbb{Z}}(\mathbb{Z}\oplus M)\cong\mathbb{Z}[t,t_1,\dots,t_\nu]$, and we denote $\Sigma^*, I:\mathbb{Z}[t,t_1,\dots,t_\nu]\to\mathbb{Z}[\mathbf{G}]\otimes_{\mathbb{Z}}\mathbb{Z}[t,t_1,\dots,t_\nu]$ respectively the co-multiplication, and the homomorphism $f\mapsto 1\otimes f$; we define $H:=\Sigma^*-I$. Similarly, for a prime p, we write $H_{\mathbb{F}_p}$ for the homomorphism induced by the co-multiplication of $G_{\mathbb{F}_p}$.

For $N:=\mathrm{Im}(H)\subset\mathbb{Z}[\mathbf{G}][t,t_1,\dots,t_\nu]$, we get the short exact sequence

$$0\longrightarrow\mathbb{Z}[t,t_1,\dots,t_\nu]^{\mathbf{G}}\xrightarrow{\ \alpha\ }\mathbb{Z}[t,t_1,\dots,t_\nu]\xrightarrow{\ H\ }N\longrightarrow 0,$$

and therefore

$$\big(\mathbb{Z}[t,t_1,\dots,t_\nu]^{\mathbf{G}}\big)\otimes_{\mathbb{Z}}\mathbb{F}_p\xrightarrow{\alpha\otimes\mathbb{F}_p}\mathbb{F}_p[t,t_1,\dots,t_\nu]\xrightarrow{H\otimes\mathbb{F}_p}N\otimes_{\mathbb{Z}}\mathbb{F}_p\longrightarrow 0 \tag{7.2}$$

is exact. We claim that the first arrow in injective: indeed, if $f\otimes\hat{1}\in\mathrm{Ker}(\alpha\otimes\mathbb{F}_p)$, then the (integral) coefficients of the polynomial f are all divisible by p. We deduce that $f=pf'$, and $f'\in\mathbb{Z}[t,t_1,\dots,t_\nu]^{\mathbf{G}}$. But then $f\otimes\hat{1}=(pf')\otimes\hat{1}=0$.

Since $\mathbb{Z}[\mathbf{G}][t,t_1,\dots,t_\nu]\otimes_{\mathbb{Z}}\mathbb{F}_p\cong\mathbb{F}_p[G_{\mathbb{F}_p}][t,t_1,\dots,t_\nu]$, the homomorphisms

$$H_{\mathbb{F}_p}\ \text{and}\ H\otimes\mathbb{F}_p:\mathbb{F}_p[t,t_1,\dots,t_\nu]\longrightarrow\mathbb{F}_p[G_{\mathbb{F}_p}][t,t_1,\dots,t_\nu]$$

coincide, so that $N_{\mathbb{F}_p}:=\mathrm{Im}(H_{\mathbb{F}_p})=N\otimes_{\mathbb{Z}}\mathbb{F}_p$. The exactness at the middle term of (7.2) implies that

$$\big(\mathbb{Z}[t,t_1,\dots,t_\nu]^{\mathbf{G}}\big)\otimes_{\mathbb{Z}}\mathbb{F}_p\xrightarrow{\ \cong\ }\mathbb{F}_p[t,t_1,\dots,t_\nu]^{G_{\mathbb{F}_p}},$$

which proves our claim. □

Proposition 7.7. *There is a prime number $p_0\in\mathbb{N}$ such that for any algebraically closed field K of characteristic zero or $\mathrm{char}\,K>p_0$ holds:*

(i) *There is $c>0$ such that $\big(Y_{\chi,K},\mathscr{L}_{Y_\chi,\chi_K^c}\big)$ is arithmetically Cohen-Macaulay;*

(ii) $H^i\big(Y_{\chi,K},\psi_K^*\mathscr{L}_{Y_\varepsilon,\varepsilon_K^n}\big)=0\qquad\forall\,n\in\mathbb{Z},\quad\forall\,i>0,\ i\neq\dim Y_{\varepsilon,K};$

$H^{\dim Y_{\varepsilon,K}}\big(Y_{\chi,K},\psi_K^*\mathscr{L}_{Y_\varepsilon,\varepsilon_K^n}\big)=0\quad\forall\,n\geqslant 0.$

We recall that *a pair* $(Y,\mathscr{A})$ consisting of a projective scheme and an ample invertible sheaf is called arithmetically Cohen-Macaulay if $\bigoplus_{n\geqslant 0}H^0\big(Y,\mathscr{A}^n\big)$ is a Cohen-Macaulay ring.

We remark that the difficulty in the second statement is to show the *simultaneous* vanishing of the cohomology groups of the $\mathscr{L}_{\chi_K^n}$'s, for variable n. For fixed n, our claim is a direct application of the upper semi-continuity theorem [14], theorem 12.8.

Proof. We choose and fix $c>0$ such that $\mathbb{Z}[\mathbf{V}]^{\mathbf{G}}_{\chi^{nc}}=\big(\mathbb{Z}[\mathbf{V}]^{\mathbf{G}}_{\chi^c}\big)^n$. Let us assume that K has characteristic zero. The Hochster-Roberts theorem [19], theorem 0.1, asserts that the ring of invariants

$$R^{G_K}=(R^{G_K})_0\oplus(R^{G_K})_+:=K\oplus\bigoplus_{n>0}K[\mathbb{V}_K]^{G_K}_{\chi_K^n}=K[\mathbb{A}^1_K\times\mathbb{V}_K]^{G_K}$$

is Cohen-Macaulay. Replacing χ by χ^c, we obtain the first statement.

The second statement is proved in two steps. If $\varepsilon_{\mathbb{C}} \in \operatorname{int.} C(\chi_{\mathbb{C}})$ then $C(\varepsilon_{\mathbb{C}}) = C(\chi_{\mathbb{C}})$, and corollary 6.3 implies that $C(\varepsilon_K) = C(\chi_K)$. Hence the quotients $Y_{\varepsilon,K}$ and $Y_{\chi,K}$ coincide, and we may assume $\varepsilon = \chi$. The Grothendieck-Serre correspondence gives the graded isomorphisms

$$\bigoplus_{n\in\mathbb{Z}} H^i\big(Y_{\chi,K}, \mathscr{O}_{Y_{\chi,K}}(n)\big) \cong H^{i+1}_{(R^{G_K})_+}(R^{G_K}), \quad \forall\, i > 0,$$

and [19], theorem 4.6, says that

$$H^i_{(R^{G_K})_+}(R^{G_K}) = 0, \text{ for } 0 < i \leq \dim Y_{\chi,K}, \quad \text{and} \quad \Big[H^{\dim Y_{\chi,K}+1}_{(R^{G_K})_+}(R^{G_K})\Big]_{\geqslant 0} = 0.$$

Our claim follows because $\mathscr{O}_{Y_{\chi,K}}(n) \cong \mathscr{L}_{\chi_K^n}$.

If $\varepsilon_{\mathbb{C}} \in \partial C(\chi_{\mathbb{C}})$ then the same holds over K. We deduce $\mathrm{R}^i\psi_{K*}\big(\psi_K^*\mathscr{L}_{\varepsilon_K^n}\big) = \mathscr{L}_{\varepsilon_K^n} \otimes \mathrm{R}^i\psi_{K*}\mathscr{O}_{Y_{\chi,K}} = 0$. Therefore $H^i\big(Y_{\chi,K}, \psi_K^*\mathscr{L}_{\varepsilon_K^n}\big) \cong H^i\big(Y_{\varepsilon,K}, \mathscr{L}_{\varepsilon_K^n}\big)$, and our claim follows from the previous step.

Now we wish to extend the results over fields of large characteristic. Since the morphism $\mathrm{pr} : \mathbf{Y}_\chi \to \operatorname{Spec}\mathbb{Z}$ is proper and flat, and $R^{G_{\mathbb{Q}}}$ is Cohen-Macaulay, [13], theorem 12.2.4 (i), implies that there is a non-empty open subset $U(\chi) \subset \operatorname{Spec}\mathbb{Z}$ such that the homogeneous rings of the fibres over $U(\chi)$ are still Cohen-Macaulay; by flat base change [25], lemma 2, R^{G_K} is Cohen-Macaulay for all fields K with $\langle \operatorname{char} K\rangle \in U(\chi)$.

The proof is divided again in two steps: if $\varepsilon_{\mathbb{C}} \in \operatorname{int.} C(\chi_{\mathbb{C}})$, we may assume as before that $\varepsilon = \chi$. The Cohen-Macaulay property implies

$$H^i\big(Y_{\chi,K}, \mathscr{L}_{\chi_K^n}\big) = 0, \quad \forall\, 0 < i < \dim Y_{\chi,K},\ \forall\, n \in \mathbb{Z}. \tag{7.3}$$

It remains to prove the vanishing of the highest cohomology group *simultaneously* for all $n \geqslant 1$. Since $\mathrm{pr} : \mathbf{Y}_\chi \to \operatorname{Spec}\mathbb{Z}$ is proper and flat, there is a positive integer $m > 0$ such that $\mathscr{O}_{Y_{\chi,\mathbb{F}_p}}$ is m-regular for all primes p. In particular

$$H^{\dim Y_{\chi,\mathbb{F}_p}}\big(Y_{\chi,\mathbb{F}_p}, \mathscr{L}_{\chi_{\mathbb{F}_p}^{nmc}}\big) = 0, \quad \forall\, n > 0. \tag{7.4}$$

We consider now the diagonal representation $\mathbf{G} \times_{\mathbb{Z}} \mathbf{G} \to \mathbf{Gl}(M \oplus M)$, and the character $\chi \times \chi^{mc} : \mathbf{G} \times_{\mathbb{Z}} \mathbf{G} \to \mathbf{G}_m$. The corresponding invariant ring is

$$R^{\mathbf{G}\times\mathbf{G}} := \mathbb{Z} \oplus \bigoplus_{n\geqslant 1} \mathbb{Z}[\mathbf{V}]^{\mathbf{G}}_{\chi^n} \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbf{V}]^{\mathbf{G}}_{\chi^{nmc}}.$$

We observe that $\operatorname{Proj} R^{\mathbf{G}\times\mathbf{G}} \cong \mathbf{Y}_\chi \times_{\mathbb{Z}} \mathbf{Y}_\chi$, and that $\big(R^{\mathbf{G}\times\mathbf{G}}[n]\big)^{\sim} \cong \mathscr{L}_{\chi^n} \boxtimes \mathscr{L}_{\chi^{nmc}}$. The Hochster-Roberts theorem implies that $R^{G_{\mathbb{Q}}\times G_{\mathbb{Q}}}$ is Cohen-Macaulay, so that $R^{G_K\times G_K}$ is still Cohen-Macaulay for $\operatorname{char} K$ large enough. Since $\dim Y_{\chi,K} < 2\dim Y_{\chi,K}$ (when the quotient is a point there is nothing to prove), we deduce on one hand that

$$H^{\dim Y_{\chi,K}}\big(Y_{\chi,K} \times Y_{\chi,K}, \mathscr{L}_{\chi^n} \boxtimes \mathscr{L}_{\chi^{nmc}}\big) = 0, \quad \forall\, n > 0$$

On the other hand, equalities (7.3) and (7.4) imply that

$$H^{\dim Y_{\chi,K}}\big(Y_{\chi,K} \times Y_{\chi,K}, \mathscr{L}_{\chi^n} \boxtimes \mathscr{L}_{\chi^{nmc}}\big) = H^{\dim Y_{\chi,K}}\big(Y_{\chi,K}, \mathscr{L}_{\chi^n}\big) \otimes_K H^0\big(Y_{\chi,K}, \mathscr{L}_{\chi^{nmc}}\big).$$

It follows that $H^{\dim Y_{\chi,K}}\big(Y_{\chi,K}, \mathscr{L}_{\chi^n}\big) = 0$, for $n > 0$. This finishes the case $\varepsilon_{\mathbb{C}} \in \operatorname{int.} C(\chi_{\mathbb{C}})$.

In general, we are going to prove that $\mathrm{R}^i\psi_{K*}\mathscr{O}_{Y_{\chi,K}} = 0$, for all $i > 0$, as soon as the characteristic of K is large enough. The conclusion follows then from the projection formula and the case treated above.

We choose a finite set $\mathcal{S} \subset \bigoplus_{n\geqslant 1}\big(\mathrm{Sym}^\bullet_{\mathbb{Z}} M\big)^{\mathbf{G}}_{\varepsilon^n}$ of homogeneous elements (for this grading), which define an open, affine covering of $\mathbf{V}^{\mathrm{ss}}(\mathbf{G},\chi)$. For $f\in\mathcal{S}$, consider the diagram

$$\begin{array}{ccccc}
\psi_{\mathbb{C}}^{-1}D(f_{\mathbb{C}},Y_{\varepsilon,\mathbb{C}}) & \longrightarrow & \psi_{\mathbb{Q}}^{-1}D(f_{\mathbb{Q}},Y_{\varepsilon,\mathbb{Q}}) & \longrightarrow & \psi_{U}^{-1}D(f,\mathbf{Y}_{\varepsilon}) \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Spec}\mathbb{C} & \longrightarrow & \operatorname{Spec}\mathbb{Q} & \longrightarrow & U
\end{array}$$

We have proved in lemma 7.4 that $H^i\big(\psi_{\mathbb{C}}^{-1}D(f_{\mathbb{C}},Y_{\varepsilon,\mathbb{C}}),\mathscr{O}_{Y_\chi,\mathbb{C}}\big)=0$, for all $i>0$. By faithfully flat base change and upper semi-continuity, there is an open subset $U(f)\subset U\subset\operatorname{Spec}\mathbb{Z}$ such that $H^i\big(\psi_{K}^{-1}D(f_K,Y_{\varepsilon,K}),\mathscr{O}_{Y_\chi,K}\big)=0$, for all $i>0$, as soon as $\langle\operatorname{char}K\rangle\in U(f)$. We let U to be the finite intersection of the $U(f)$'s obtained this way. Applying [14], proposition 9.3 and proposition 8.5, we deduce the vanishing of $\mathrm{R}^i\psi_{K*}\mathscr{O}_{Y_\chi,K}|_{D(f_K,Y_{\varepsilon,K})}$, for all $i>0$ and $f\in\mathcal{S}$, as soon as $\langle\operatorname{char}K\rangle\in U$. Since $\big\{D(f_K,Y_{\varepsilon,K})\big\}_{f\in\mathcal{S}}$ is an open, affine covering of $Y_{\varepsilon,K}$, we deduce that the higher direct images $\mathrm{R}^i\psi_{K*}\mathscr{O}_{Y_\chi,K}$, $i>0$, vanish. □

The next step is to prove the vanishing of the higher cohomology for nef line bundles. Consider $\varepsilon_{\mathbb{C}}\in\partial C(\chi_{\mathbb{C}})\cap\mathcal{X}^*(G_{\mathbb{C}})$, and assume $\mathscr{L}_{Y_\chi,\varepsilon_{\mathbb{C}}}\to Y_{\chi,\mathbb{C}}$ is invertible. Then the same holds over some non-empty, open subset $U'(\varepsilon)\subset\operatorname{Spec}\mathbb{Z}$. We define the representation

$$\bar\rho:\bar{\mathbf{G}}:=\mathbf{G}\times_{\operatorname{Spec}\mathbb{Z}}\mathbf{G}_m\longrightarrow\mathbf{Gl}\big(M\oplus\mathbb{Z}^{\oplus 2}\big),\quad \bar\rho(g,t):=\operatorname{diag}\big(\rho(g),t\varepsilon^{-1}(g),t\big),$$

and notice that since $\big(\mathrm{Sym}^\bullet_{\mathbb{Z}} M\big)^{\mathbf{G}}=\mathbb{Z}$, $\big(\mathrm{Sym}^\bullet_{\mathbb{Z}}(M\oplus\mathbb{Z}^2)\big)^{\bar{\mathbf{G}}}=\mathbb{Z}$ too. After replacing χ with a suitably large multiple, $\big(\mathrm{Sym}^\bullet_{\mathbb{Z}} M\big)^{\mathbf{G},\chi}$ becomes generated in degree one. We define

$$\bar\chi:\bar{\mathbf{G}}\longrightarrow\mathbf{G}_m,\quad \bar\chi(g,t):=\chi(g)t,$$

and observe that

$$\begin{aligned}\big(\mathrm{Sym}^\bullet_{\mathbb{Z}}(M\oplus\mathbb{Z}^2)\big)^{\bar{\mathbf{G}},\bar\chi} &= \big((\mathrm{Sym}^\bullet_{\mathbb{Z}} M)[w_1,w_2]\big)^{\bar{\mathbf{G}},\bar\chi}=\bigoplus_{n\geqslant 0}\big((\mathrm{Sym}^\bullet_{\mathbb{Z}} M)[w_1,w_2]\big)^{\bar{\mathbf{G}}}_{\bar\chi^n}\\ &=\bigoplus_{n\geqslant 0}\ \bigoplus_{a+b=n}(\mathrm{Sym}^\bullet_{\mathbb{Z}} M)^{\mathbf{G}}_{\chi^n\varepsilon^a}w_1^a w_2^b.\end{aligned}\tag{7.5}$$

By corollary 6.3, there is an open subset $U''(\varepsilon)\subset\operatorname{Spec}\mathbb{Z}$ such that $\varepsilon_K\in C(\chi_K)$ for any algebraically closed field K with $\langle\operatorname{char}K\rangle\in U''(\varepsilon)$. We define $U(\varepsilon):=U'(\varepsilon)\cap U''(\varepsilon)$. Then $U(\varepsilon)=\operatorname{Spec}B$, where B is obtained from $\mathbb{Z}$ by inverting finitely many primes.

Lemma 7.8. *Let $\varepsilon:\mathbf{G}\to\mathbf{G}_m$ be a character such that $\varepsilon_{\mathbb{C}}\in C(\chi_{\mathbb{C}})\cap\mathcal{X}^*(G_{\mathbb{C}})$, and moreover $\mathscr{L}_{Y_\chi,\varepsilon_{\mathbb{C}}}\to Y_{\chi,\mathbb{C}}$ is invertible. Then there is a non-empty, open subset $U(\varepsilon)\subset\operatorname{Spec}\mathbb{Z}$ over which the following isomorphism holds*

$$\mathbb{P}\big(\mathscr{O}_{\mathbf{Y}_\chi}\oplus\mathscr{L}_{\mathbf{Y}_\chi,\varepsilon}\big)|_{U(\varepsilon)}:=\underline{\mathrm{Proj}}\big(\mathrm{Sym}^\bullet_{\mathscr{O}_{\mathbf{Y}_\chi}}(\mathscr{O}_{\mathbf{Y}_\chi}\oplus\mathscr{L}_{\mathbf{Y}_\chi,\varepsilon})\big)\cong\mathrm{Proj}\big(\mathbb{Z}[\mathbf{V}\times_{\mathbb{Z}}\mathbb{A}^2_{\mathbb{Z}}]^{\bar{\mathbf{G}},\bar\chi}\big)|_{U(\varepsilon)}.$$

Proof. Is a straightforward computation. □

The main cohomology vanishing result is the following:

Theorem 7.9. *Let $\mathbf{G}\to\mathbf{Gl}(M)$ be a representation such that $(\mathrm{Sym}^\bullet_{\mathbb{Z}} M)^{\mathbf{G}}=\mathbb{Z}$, and consider two characters χ,ε of $\mathbf{G}$ with the properties:*
- $\operatorname{codim}_{\mathbb{V}_{\mathbb{C}}}\mathbb{V}^{\mathrm{us}}(G_{\mathbb{C}},\chi_{\mathbb{C}})\geqslant 2$*;*
- $\mathscr{L}_{Y_\chi,\varepsilon_{\mathbb{C}}}\longrightarrow Y_{\chi,\mathbb{C}}$ *is invertible and nef.*

Then there is a prime p_0 such that for all algebraically closed fields of characteristic $\operatorname{char}K>p_0$ *holds:* $H^i\big(Y_{\chi,K},\mathscr{L}^{\otimes n}_{Y_\chi,\varepsilon_K}\big)=0$, $\forall\, n\geqslant 0$, $\forall\, i>0$.

Proof. Let $U(\varepsilon) \subset \operatorname{Spec}\mathbb{Z}$ be as above, and let $\mathscr{L} := \mathscr{L}_{Y_\chi,\varepsilon}$. Then $\mathscr{L}|_{U(\varepsilon)} \to \mathbf{Y}_\chi|_{U(\varepsilon)}$ is relatively nef, and therefore $\mathscr{O}_{\mathbb{P}(\mathscr{O}\oplus\mathscr{L})}(1)|_{U(\varepsilon)} \longrightarrow \mathbb{P}\big(\mathscr{O}\oplus\mathscr{L}\big)|_{U(\varepsilon)}$ is still relatively nef.

Proposition 7.7, together with lemmata 7.5 and 7.8, imply that there is a constant $c > 0$ having the property that, after possibly shrinking $U(\varepsilon)$ further,

$$H^i\big(\mathbb{P}\big(\mathscr{O}\oplus\mathscr{L}\big)_K, \mathscr{O}(1)^{nc}_K\big) = 0 \;\forall\, n \geqslant 0,$$

on all geometric fibres over $U(\varepsilon)$. Denoting $\pi : \mathbb{P}\big(\mathscr{O}\oplus\mathscr{L}\big) \to \mathbf{Y}_\chi$ the natural projection, we have $\mathrm{R}^i(\pi_K)_*\mathscr{O}(1)^{nc}_K = 0$, $\forall\, i > 0\;\forall\, n \geqslant 0$, and therefore

$$0 = H^i\big(\mathbb{P}\big(\mathscr{O}\oplus\mathscr{L}\big)_K, \mathscr{O}(1)^{nc}_K\big) = H^i\big(Y_{\chi,K}, (\pi_K)_*\mathscr{O}(1)^{nc}_K\big) = \bigoplus_{a=0}^{nc} H^i\big(Y_{\chi,K}, \mathscr{L}^a_K\big).$$

Taking $n > 0$ arbitrarily large, we obtain the cohomology vanishing result. □

Remark 7.10. (i) It is actually possible 'to squeeze out' some more information, namely that the prime p_0 appearing in the previous theorem can be chosen *independent* of ε. In other words, one is able to find a prime $p(Y_\mathbb{Q})$, depending only on the quotient $Y_\mathbb{Q} := Y_{\chi,\mathbb{Q}}$, for which one has vanishing of the higher cohomology groups, for all invertible, nef sheaves, in characteristic larger than $p(Y_\mathbb{Q})$.

This can be seen as follows: consider a *finite* set of characters $\{\varepsilon_j\}_{j\in J}$ such that $\{\mathscr{L}_{\varepsilon_j}\}_{j\in J}$ generate the nef cone $C(\chi_\mathbb{Q})$ of $Y_{\chi,\mathbb{Q}}$ over $\mathbb{Z}_{\geqslant 0}$; then the same holds over a non-empty, open subset of $\operatorname{Spec}\mathbb{Z}$. Similarly as in lemma 7.8, one proves that

$$\mathbb{P}\big(\mathscr{O}\oplus\bigoplus_{j\in J}\mathscr{L}_j\big) \;\cong \big[\mathbf{V}\times_\mathbb{Z}\underbrace{\mathbb{A}^2_\mathbb{Z}\times_\mathbb{Z}\ldots\times_\mathbb{Z}\mathbb{A}^2_\mathbb{Z}}_{|J|\text{ times}}\big]/\!\!/\mathbf{G}\times_\mathbb{Z}\mathbf{G}_m^{|J|}$$

for a suitable action and linearization. Applying now proposition 7.7 to large multiples of $\mathscr{O}_{\mathbb{P}(\mathscr{O}\oplus\bigoplus_{j\in J}\mathscr{L}_j)}(1)$, just as in 7.9, one obtains the desired result.

(ii) Combining the observation above with proposition 7.7, we deduce that for fields K whose characteristic is either zero or larger than $p(Y_\mathbb{Q})$, $Y_{\chi,K}$ is arithmetically Cohen-Macaulay with respect to any ample line bundle on it (see [17], proposition 14.1, for the passage from a line bundle to a positive multiple of it).

Remark 7.11. M. Brion brought to my attention the papers [2, 27], which both address the issue that 'quantization commutes with reduction': [27] adopts the algebraic viewpoint, while [2] uses analytic methods. However, to our best knowledge, the cited papers treat neither the case of nef bundles nor that of mixed characteristic.

8. A stability property

It is well known that the tangent bundle of the projective space is stable, or more generally that the tautological bundles over a Grassmann variety are stable. The goal of this section is to generalize these results to our setting in characteristic zero.

We assume throughout this section that the ring of invariants $K[\mathbb{V}]^G = K$. We decompose

$$V = \bigoplus_{\omega\in\mathcal{X}} V_\omega{}^{\oplus\nu_\omega}, \quad \text{with } \mathcal{X}\subset\mathcal{X}^*(T),$$

into its G-isotypical components, with the corresponding multiplicities. The main result of this section is:

Theorem 8.1. *Assume $\chi \in \mathcal{X}^*(G)$ has the property that G acts freely on $\Omega := \mathbb{V}^{\rm ss}(G,\chi)$, and let Y be the quotient. For $\omega \in \mathcal{X}$ we denote by $\mathcal{F}_\omega := \Omega(V_\omega)$ the associated vector bundle over Y. Assume that $\nu_\omega \geqslant \dim V_\omega$ holds for all $\omega \in \mathcal{X}$. Then the vector bundles $\mathcal{F}_\omega \to Y$, $\omega \in \mathcal{X}$, are slope semi-stable with respect to the polarization defined by the character χ.*

First we prove an auxiliary result of (possibly) independent interest. We denote $\{G_j\}_{j\in J}$ the simple factors of G, and for each j we denote $\gamma_j : G \to G_j$ be the corresponding quotient morphism. Using the γ_j's we extend the structural group of $\Omega \to Y$, and obtain the principal G_j-bundles $\Omega(G_j) \to Y$. We first investigate their stability.

Proposition 8.2. *Assume $\chi \in \mathcal{X}^*(G)$ has the property that G acts freely on $\Omega := \mathbb{V}^{\rm ss}(G,\chi)$, and let Y be the quotient. Assume that $\nu_\omega \geqslant \dim V_\omega$ holds for all $\omega \in \mathcal{X}$. Then the principal G_j-bundles $\Omega(G_j) \to Y$, $j \in J$, obtained by extending the structural group are semi-stable.*

Proof. We fix $j \in J$, and a maximal parabolic subgroup $P_j \subset G_j$; we let $P := \gamma_j^{-1} P_j$: it is a maximal parabolic subgroup of G. Observe that in this case the associated homogeneous bundles $\big(\Omega(G_j)\big)(G_j/P_j)$ and $\Omega\big(G/P\big)$ are isomorphic.

We denote $H = \prod\limits_\omega H_\omega := \prod\limits_\theta \mathrm{Gl}_K(\nu_\omega)$: it acts naturally on $\mathbb{V}$, and the G- and H-actions on $\mathbb{V}$ commute. It follows that H still acts on $\Omega(G/P)$ by

$$H \times \Omega(G/P) \longrightarrow \Omega(G/P), \quad h \times [y, gP] := [hy, gP].$$

We must prove that for any reduction of the structural group

$$s : Y^o \to \big(\Omega(G_j)\big)(G_j/P_j) = \Omega\big(G/P\big), \quad \text{with } Y^o \subset Y \text{ open and } \mathrm{codim}_Y(Y \setminus Y^o) \geqslant 2,$$

we have $\deg_Y\big(s^*\mathsf{T}_{\Omega(G/P)/Y}\big) \geqslant 0$. The idea is to move s using the H-action on $\Omega(G/P)$. Let $\hat{y} \in Y$ be a generic point, and consider $y \in \Omega$ over $\hat{y}$. We define the following subgroups

$$H_{\hat{y}} := \{h \in H \mid \exists\, g_h \in G \text{ s.t. } hy = \rho(g_h^{-1})y\} = \prod_\theta H_{\theta,\hat{y}}, \quad \text{and} \quad H_y := \mathrm{Stab}_H(y)$$

of H. Since G acts freely on Ω, the asignment $h \mapsto g_h$ defines a group homomorphism $\rho_{\hat{y}} : H_{\hat{y}} \to G$ whose kernel is H_y.

<u>Claim</u> $H_{\hat{y}}/H_y \to G/Z(G)^\circ$ is surjective. Write $y = (y_\omega)_\omega$ with respect to the direct sum decomposition of V; for each $\omega \in \mathcal{X}$, $y_\omega = (y_{\omega 1}, \ldots, y_{\omega m_\omega})$. Since $y \in \Omega$ is chosen generically, and $m_\omega \geqslant \dim V_\omega =: d_\omega$, we may assume that for each $\omega \in \mathcal{X}$ the vectors $y_{\omega 1}, \ldots, y_{\omega m_\omega}$ span V_ω. Equivalently, we may view y_ω as a surjective homomorphism $K^{\nu_\omega} \to V_\omega$.

For $g \in G$ holds $\rho(g)y = (\rho_\omega(g)y_\omega)_\omega$. Using that $m_\omega \geqslant d_\omega$, we deduce that for each $\omega \in \mathcal{X}$ there is $h_\omega \in \mathrm{Gl}_K(m_\omega)$ such that $h_\omega y_\omega = \rho_\omega(g^{-1})y_\omega$. For $h := (h_\omega)_\omega$ we have $hy = \rho(g^{-1})y$, that is $g \in \mathrm{Image}\big(H_{\hat{y}}/H_y \to G\big)$. This proves our claim.

Back to the proof of our proposition. We observe that the infinitesimal action of H_y preserves the restriction to the fibre $q^{-1}(\hat{y}) = \{[y, gP] \mid g \in G\} \cong G/P$ of the relative tangent bundle $\mathsf{T}_{\Omega(G/P)/Y}$. By this isomorphism the relative tangent bundle corresponds to $\mathsf{T}_{G/P} \to G/P$.

The previous claim implies that the infinitesimal action $\mathcal{L}ie(H_y) \to \mathsf{T}_{\Omega(G/P)/Y, s(\hat{y})}$ is surjective. Therefore we can produce a section $\sigma \in H^0(Y^o, s^*\det\mathsf{T}_{\Omega(G/P)/Y})$ which does not vanish at the point $\hat{y}$. This proves that $\deg_Y\big(s^*\mathsf{T}_{\Omega(G/P)/Y}\big) \geqslant 0$. □

Proof. (Theorem 8.1) We may assume that $G = Z(G)^\circ \times \big(\times_{j\in J} G_j\big)$. Since each $\Omega(G_j)$ is semi-stable, $\Omega \to Y$ itself is semi-stable. The homomorphism $\rho_\omega : G \to \mathrm{Gl}(V_\omega)$ maps $Z(G)^\circ$

into the center of $\mathrm{Gl}(V_\omega)$. Using [23], theorem 3.18, we deduce that the associated vector bundle $\mathcal{F}_\omega = \Omega(V_\omega) \to Y$ is semi-stable. □

Example 8.3. We wish apply these ideas to the construction of algebro-geometric approximations of classifying spaces *cf.* [8]. The basic example, which we are going to generalize, is that of the classifying space for $\mathrm{Gl}(n)$. This is the infinite Grassmannian of d-dimensional quotients, obtained as the inductive limit $\lim_{m\to\infty} \mathrm{Grass}(m,d)$. Its nice feature is that the universal quotient bundles $\mathcal{Q}_m \to \mathrm{Grass}(m,d)$ are stable for all m.
Claim Let V be a non-zero G-module such that $\mathbb{C}[\mathbb{V}]^T = \mathbb{C}$. Then holds:
(i) There is a 1-PS $\lambda \in \mathcal{X}_*(T)$ such that its weight on V are all strictly positive.
(ii) G is not semi-simple.

Proof. (i) Let Φ be the weights of the T-module V. Then the set of weights of the T on $\mathbb{C}[\mathbb{V}]$ is the 'cone' $\sum_{\eta\in\Phi} \mathbb{N}\eta$. Since $\mathbb{C}[\mathbb{V}]^T = \mathbb{C}$, this cone is strictly convex, so that there is $\lambda \in \mathcal{X}_*(T)$ with $\langle \eta, \lambda\rangle > 0$ for all $\eta \in \Phi$.

(ii) Assume G is semi-simple. The previous step implies that $\mathbb{C}[\mathbb{V}^m]^T = \mathbb{C}$, and therefore $\mathbb{C}[\mathbb{V}^m]^G = \mathbb{C}$ for all $m \geqslant 1$. Since G is semi-simple, it follows from [21] that G has an open orbit in $\mathbb{V}^m$. For m large we get a contradiction. □

Let G be a connected, reductive group which is not semi-simple; then $G = (Z \times G')/F$, where $Z := Z(G)^\circ$ is positive dimensional and F is a finite group. Consider moreover a faithful representation $\rho : G \to \mathrm{Gl}(V)$. Twisting ρ with a suitable 'large' character of Z, we may assume that $\mathbb{C}[\mathbb{V}]^T = \mathbb{C}$, and $\chi := \det(\rho)$ is not in a wall for the G-action on the $\mathbb{V}^m$'s, simultaneously for all m (see section 3). Then $\mathbb{C}[\mathbb{V}^m]^G = \mathbb{C}$ for all $m \geqslant 1$, and for m sufficiently large the multiplicity condition in proposition 8.2 is fulfilled.

The inclusion $\mathbb{V}^m \hookrightarrow \mathbb{V}^{m+1}$, $x \mapsto (x, 0)$, restricts to $\Omega_m := (\mathbb{V}^m)^{\mathrm{ss}}(G,\chi) \hookrightarrow \Omega_{m+1} := (\mathbb{V}^{m+1})^{\mathrm{ss}}(G,\chi)$, which induces the morphisms $Y_m \to Y_{m+1}$ at the quotient level. Since the quotients $\Omega_m \to Y_m$ are geometric, $Y_m \to Y_{m+1}$ is a closed embedding.

Assume moreover that G acts freely on the Ω_m's, for m large enough. What we obtain is a directed system $\Omega_m \to Y_m$ of *semi-stable* principal G-bundles over projective bases. Since the codimension of the unstable locus $(\mathbb{V}^m)^{\mathrm{us}}(G,\chi) \hookrightarrow \mathbb{V}^m$ grows at least linearly with m, these principal bundles form algebro-geometric substitutes for $\mathrm{EG} \to \mathrm{BG}$.

In the case $G = \mathrm{Gl}(d)$ and V is its standard representation, this construction yields the Grassmannian $\mathrm{Grass}(m,d)$ together with the universal quotient bundle.

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