

INTRODUCTION TO LIE GROUPS

Exercise sheet 5

(due on Monday, February 6th)

Exercise 1. Let G be a connected Lie group with Lie algebra \mathfrak{g} , $f : G \rightarrow H$ a Lie group homomorphism. The $\mathfrak{S}(f)$ is a Lie subgroup of H whose Lie algebra is $\mathfrak{S}(df_e)$.

Exercise 2. Show that given any compact topological group G and any representation $\rho : G \rightarrow \text{GL}(V)$, there exists a positive definite inner product on V which is G -invariant.

Exercise 3. Let G be a compact second countable group and $b : G \times G \rightarrow \mathbb{R}$ be a real valued continuous map such that

- (1) $b(1, 1) = 0$, and
- (2) $b(xy, z) - b(y, z) + b(x, y) - b(x, yz) = 0$ for all $x, y, z \in G$.

Show that there exists a continuous function $\alpha : G \rightarrow \mathbb{R}$ such that

- (i) $\alpha(1) = 0$,
- (ii) $b(x, y) = \alpha(xy) - \alpha(x) - \alpha(y)$ for all $x, y \in G$.

Exercise 4. Let G be a connected locally compact second countable group, $Z < G$ a discrete central subgroup and assume that G/Z is compact. The goal of this exercise is to show that any homomorphism $\varphi : Z \rightarrow \mathbb{R}^+$ extends to a continuous homomorphism $\psi : G \rightarrow \mathbb{R}^+$. The proof will consist of the following two steps:

- (i) Show that there exists a continuous function $f : G \rightarrow \mathbb{R}^+$ such that $f|_Z = \varphi$ and $f(xz) = f(x)\varphi(z)$ for all $x \in G$ and $z \in Z$.
(*Hint:* Choose an appropriate compact subset K of G such that $G = ZK$ and define $f_1(x) := \sum_{z \in Z} g(xz)\varphi(z)^{-1}$, where G is a nonnegative continuous function on G with compact support which is the identity on K .)
- (ii) Show that there exists a continuous homomorphism $\psi : G \rightarrow \mathbb{R}^+$ such that $\psi|_Z = \varphi$.
(*Hint:* Define $h : G \times G \rightarrow \mathbb{R}$ by $h(x, y) := \ln f(x, y) - \ln f(x) - \ln f(y)$ and use Exercise 3.)

Exercise 5. Let G be, as above, any locally compact second countable group and let $Z < G$ be any discrete central subgroup such that G/Z is compact.

(i) Show that Z is finitely generated.

(*Hint:* Choose an appropriate compact set $K < G$ such that $G = ZK^\circ$, where K° is the interior of K . Choose $z_1, \dots, z_n \in Z$ such that $KK^{-1} \subseteq \cup_i^n z_i K^\circ$ and consider Z_1 the group generated by z_1, \dots, z_n . Show that $G = Z_1 K$ and that Z_1 is of finite index in Z .)

(ii) Use Exercise 4 to show that if G has no continuous homomorphisms into \mathbb{R}^+ then it is compact.

(iii) Deduce that the universal covering of a compact semisimple Lie group is also compact.

Exercise 6. Let \mathfrak{g} be the Lie algebra over \mathbb{C} with basis $\{H, E, F\}$ and bracket

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.$$

(i) Find a basis of the realification $\mathfrak{g}_{\mathbb{R}}$ of \mathfrak{g} and the corresponding bracket induced by the bracket of \mathfrak{g} .

(ii) Determine a real form \mathfrak{h} of \mathfrak{g} , that is a real subalgebra \mathfrak{h} of $\mathfrak{g}_{\mathbb{R}}$ such that its complexification $\mathfrak{h}^{\mathbb{C}}$ is isomorphic to \mathfrak{g} as a complex algebra.

(iii) Show that \mathfrak{g} is simple.

(iv) Show that $(\mathfrak{g}_{\mathbb{R}})^{\mathbb{C}}$ is not simple, that is find ideals \mathfrak{k} and $\bar{\mathfrak{k}}$ such that $(\mathfrak{g}_{\mathbb{R}})^{\mathbb{C}} = \mathfrak{k} \oplus \bar{\mathfrak{k}}$. Determine isomorphisms $\phi : \mathfrak{k} \rightarrow \mathfrak{g}$ and $\bar{\phi} : \bar{\mathfrak{k}} \rightarrow \mathfrak{g}^*$, where \mathfrak{g}^* is the adjoint algebra of \mathfrak{g} .