PRINCIPAL COVARIANTS, MULTIPLICITY-FREE ACTIONS, AND THE K-TYPES OF HOLOMORPHIC DISCRETE SERIES

Roger Howe and Hanspeter Kraft

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Abstract. We prove a result on the structure of the K-types for holomorphic discrete series of $\mathrm{Sp}(2n,\mathbb{R})$. The proof applies the theory of multiplicity-free actions to the realization of holomorphic discrete series by means of the dual pair $(\mathrm{Sp}_{2n}, \mathrm{O}_m)$.

A major method for studying representations of a semisimple Lie group G has been to consider the decomposition of such a representation under restriction to the maximal compact subgroup K of G. One of Harish-Chandra's early basic results was that the multiplicity of an irreducible representation of K (a *K-type*) in the restriction of an irreducible representation of G is finite, in fact bounded by the dimension of the K-type ([Kna], [Wal], [War]). Formulas for the multiplicities of K-types are known in many cases. In particular, the method of cohomological induction produces attractive formulas for multiplicities of K-types, expressed as alternating sums over some Weyl group ([Kna], [KnVo], [Wal]). A well-known problem with this kind of formula is that it is hard to use to answer practical questions, for example, "Is the multiplicity zero or positive?". Thus for understanding K-types, it is desirable to supplement the alternating sum formulas with other information. The main goal of this paper is to provide such supplementary information for holomorphic discrete series representations.

The holomorphic discrete series were first constructed by Harish-Chandra in an early essay toward his Plancherel Formula [HC IV-VI]. An alternating sum formula for K-types of holomorphic discrete series was established in [Sch1], [Sch2]. On the other hand, one has a description of a holomorphic discrete series representation as a generalized Verma module ([Kna], [Sch1,2]). This description reveals a good deal of information about the K-structure of such representations. We recall it.

Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K respectively, and let $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{k}_{\mathbb{C}}$ denote their complexifications. We have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}, \tag{1}$$

where $\mathfrak{p}$ is the orthogonal complement to $\mathfrak{k}$ with respect to the Killing form on $\mathfrak{g}$. When G allows holomorphic discrete series, $\mathfrak{k}$ has a non-trivial center, and on complexification, we get

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{p}^{+} + \mathfrak{p}^{-}, \tag{2}$$

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where $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are the eigenspaces for the center of $\mathfrak{k}$ acting on $\mathfrak{p}_{\mathbb{C}}$ by the adjoint representation. Each of $\mathfrak{p}^+$ and $\mathfrak{p}^-$ is an abelian Lie algebra. Hence the universal enveloping algebra of $\mathfrak{p}^+$ is isomorphic to the symmetric algebra $\mathcal{S}(\mathfrak{p}^+)$. The adjoint action of K on $\mathfrak{p}^+$ makes $\mathcal{S}(\mathfrak{p}^+)$ into a K-module. The K-module structure of $\mathcal{S}(\mathfrak{p}^+)$ is very well understood [Jo],[Sch1] [GW]. It is multiplicity-free, that is, each irreducible representation of K occurs in $\mathcal{S}(\mathfrak{p}^+)$ at most once. Furthermore (and as a consequence of multiplicity-freeness; see for example [Ho]), if one fixes a maximal unipotent subalgebra $\mathfrak{u}_K$ of $\mathfrak{k}_{\mathbb{C}}$, then the algebra $\mathcal{S}(\mathfrak{p}^+))^{\mathfrak{u}_K}$ of $\mathfrak{u}_K$ invariant vectors in $\mathcal{S}(\mathfrak{p}^+)$ (the highest weight vectors with respect to $\mathfrak{u}_K$), is a polynomial algebra on certain canonical and explicitly known generators.

Let ρ be a holomorphic discrete series representation of G, and let V be the associated $(\mathfrak{g}, K)$-module of K-finite vectors [Kna], [Wal]. The first key property of holomorphic discrete series is that the subspace of V consisting of vectors annihilated by the elements of $\mathfrak{p}^-$ defines an irreducible representation for K ([HC], [Kna], [Sch1,2]). This is called the *lowest K-type* for ρ. Denote the lowest K-type by μ_ρ. Then the structure of all of V as a K-module is given by taking a tensor product with $\mathcal{S}(\mathfrak{p}^+)$:

$$\rho_{|K} \simeq \mu_\rho \otimes \mathcal{S}(\mathfrak{p}^+). \tag{3}$$

The space $V^{\mathfrak{u}_K}$ of K-highest weight vectors in V will be a module for $\mathcal{S}(\mathfrak{p}^+)^{\mathfrak{u}_K}$, and knowledge of $V^{\mathfrak{u}_K}$ as an $\mathcal{S}(\mathfrak{p}^+)^{\mathfrak{u}_K}$-module implies understanding of V as a K-module. Thus it is nice to know that the $\mathcal{S}(\mathfrak{p}^+)^{\mathfrak{u}_K}$-module structure of $V^{\mathfrak{u}_K}$ is as simple as it could be.

Theorem 1. *For $G = \mathrm{Sp}(2n, \mathbb{R})$, the real symplectic group in $2n$ variables, the space $V^{\mathfrak{u}_K}$ of K-highest weight vectors in the holomorphic discrete series representation V is a free module for $\mathcal{S}(\mathfrak{p}^+)^{\mathfrak{u}_K}$.*

Remark. It can be shown by a counting argument that the rank of $V^{\mathfrak{u}_K}$ over $\mathcal{S}(\mathfrak{p}^+)^{\mathfrak{u}_K}$ must be $\dim \mu_\rho$.
There is no reason to expect this result to fail for other groups which have holomorphic discrete series. However, our method of proof does not apply to the general case. It relies on the realization of the holomorphic discrete series in the local theta correspondence for the dual pair $(\mathrm{O}_m, \mathrm{Sp}(2n, \mathbb{R}))$ [Ge], [KaVe], [Sa]. A similar proof works for the groups $\mathrm{O}^*(2m)$ and $\mathrm{U}(p, q)$. It would be interesting to construct an argument valid for all groups whose associated symmetric space is hermitian.

Our approach to Theorem 1 uses the realization of holomorphic discrete series associated to the reductive dual pair $(\mathrm{O}_m, \mathrm{Sp}(2n, \mathbb{R}))$, with $m \geq 2n$. (See [Ge], [GrK], [KaVe], and [Sa]; we note that these correspondences were among the earliest examples of dual pair correspondences.) This gives a coherent realization of all holomorphic discrete series in the same vector space, which may be realized as a Fock space of holomorphic functions on $\mathbb{C}^{mn}$. The $\mathrm{O}_m \times \mathrm{U}_n$-finite functions then appear as polynomial functions, and we can reduce the statement of the theorem to a result in invariant theory. We can then use the theory of multiplicity-free actions [Ho], [Krf], [Krm], [Kno] and the geometry of the action of $K_{\mathbb{C}}$ on complex flag manifolds to establish the desired result.

Let O_m denote the orthogonal group in m variables. Denote by $\mathbb{C}^{mn}$ the space of $m \times n$ matrices, and let $\mathrm{O}_m(\mathbb{C})$ act on $\mathbb{C}^{mn}$ by matrix multiplication on the left. Let GL $(\mathbb{C})$ act on $\mathbb{C}^{mn}$ by multiplication on the right. These actions commute with each other. We can extend them to actions on the polynomial functions $\mathcal{P}(\mathbb{C}^{mn})$ in the usual way. $\mathrm{O}_m(\mathbb{C})$ is the complexification of $\mathrm{O}_m = \mathrm{O}_m(\mathbb{R})$ and the natural action of $\mathrm{O}_m(\mathbb{C})$ on $\mathcal{P}(\mathbb{C}^{mn})$ may be identified to the holomorphic extension to $\mathrm{O}_m(\mathbb{C})$ of the action of O_m on Fock space. Similarly, $\mathrm{GL}_n(\mathbb{C})$ is the complexification of U_n. The natural action of $\mathrm{GL}_n(\mathbb{C})$ on $\mathcal{P}(\mathbb{C}^{mn})$ is not quite, but almost the holomorphic extension to $\mathrm{GL}_n(\mathbb{C})$ of the action of U_n on Fock space. To get the holomorphic extension exactly, we should twist the natural action of $\mathrm{GL}_n(\mathbb{C})$ by $(\det)^{m/2}$. (See formulas (4b) below for the infinitesimal action.)

Denote by r_{ij}^2 the inner product of the i-th and j-th column of an element of $\mathbb{C}^{mn}$. The r_{ij}^2 are quadratic polynomials on $\mathbb{C}^{mn}$ and they are invariant under the action of O_m. The First Fundamental Theorem of Classical Invariant Theory (see [Ho], [We]) for O_m says that the r_{ij}^2 generate the full algebra $\mathcal{P}(\mathbb{C}^{mn})^{\mathrm{O}_m}$ of polynomials invariant under O_m. Denote by Δ_{ij} the "partial Laplacian" corresponding to the i-th and j-th rows of $\mathbb{C}^{mn}$. Explicitly, if $\{z_{ij} \mid 1 \le i \le m\}$ are the coordinates on the j-th copy of $\mathbb{C}^m$ with respect to a fixed orthonormal basis, then

$$\text{(4a)} \qquad r_{ij}^2 = \sum_{a=1}^{m} z_{ai} z_{aj} \quad \text{and} \quad \Delta_{ij} = \sum_{a=1}^{m} \frac{\partial^2}{\partial z_{ai} \partial z_{aj}}, \qquad i, j = 1, \ldots, n.$$

Denote by $\mathfrak{sp}^{(2,0)}$ the linear span of the r_{ij}^2 considered as operators on $\mathcal{P}(\mathbb{C}^{mn})$, and denote by $\mathfrak{sp}^{(0,2)}$ the span of the Δ_{ij}. Denote by $\mathfrak{sp}^{(1,1)}$ the span of the commutators $[\Delta_{ij}, r_{k\ell}^2]$. Then, $\mathfrak{sp}^{(1,1)}$ is a Lie algebra—it is a very mild perturbation of the Lie algebra defined by the infinitesimal action of the Lie algebra $\mathfrak{gl}_n$ of $\mathrm{GL}(n, \mathbb{C})$ acting on $\mathbb{C}^{mn}$ by multiplication on the right. Precisely, it is spanned by operators

$$\text{(4b)} \qquad \tilde{E}_{ij} = \left(\sum_{a=1}^{m} z_{ai} \frac{\partial}{\partial z_{aj}} \right) + \delta_{ij} \frac{m}{2} = E_{ij} + \delta_{ij} \frac{m}{2},$$

where the E_{ij} are the standard "polarization operators," which are the infinitesimal generators of standard action of GL_n. Then

$$\text{(4c)} \qquad \mathfrak{sp}_{\mathbb{C}} = \mathfrak{sp}^{(1,1)} \oplus \mathfrak{sp}^{(0,2)} \oplus \mathfrak{sp}^{(2,0)}$$

is a Lie algebra isomorphic to $\mathfrak{sp}(2n, \mathbb{C})$, and decomposition (4c) is an instance of the complexified Cartan decomposition (2), with $\mathfrak{sp}^{(1,1)}$ playing the role of $\mathfrak{k}_{\mathbb{C}}$, and $\mathfrak{sp}^{(2,0)}$ being $\mathfrak{p}^+$, and $\mathfrak{sp}^{(0,2)}$ being $\mathfrak{p}^-$. The operators spanning $\mathfrak{sp}_{\mathbb{C}}$ commute with the action of O_m on $\mathcal{P}(\mathbb{C}^{mn})$. The joint action of O_m and $\mathfrak{sp}_{\mathbb{C}}$ on $\mathcal{P}(\mathbb{C}^{mn})$ gives rise to a decomposition ([KaVe], [Ho])

$$\begin{aligned} \text{(5)} \qquad \mathcal{P}(\mathbb{C}^{mn}) &= \sum_{\tau} \mathcal{P}(\mathbb{C}^{mn})^{\mathrm{O}_m, \tau} = \sum_{\tau} H_\tau \cdot \mathbb{C}[r_{ij}^2] \\ &= \sum_{\tau} (\tau \otimes \tilde{\tau}) \cdot \mathbb{C}[r_{ij}^2] = \sum_{\tau} \tau \otimes V_\tau . \end{aligned}$$

The variable of summation in formulas (5) runs over the set of irreducible representations τ of O_m, and each summand $\mathcal{P}(\mathbb{C}^{mn})^{\mathrm{O}_m, \tau}$ is the τ-isotypic component for the action of O_m on $\mathcal{P}(\mathbb{C}^{mn})$. (Some summands may be trivial.) The three last summations express three different aspects of the structure of these isotypic components. The space H_τ is the space of τ-isotypic harmonics, consisting of elements in

$\mathcal{P}(\mathbb{C}^{mn})^{\mathrm{O}_m,\tau}$ which are annihilated by $\mathfrak{sp}^{(0,2)}$, i.e., by all the partial Laplacians Δ_{ij}. The second summation expresses the well-known general fact that $\mathcal{P}(\mathbb{C}^{mn})^{\mathrm{O}_m,\tau}$ is generated as a $\mathbb{C}(r_{ij}^2)$-module by H_τ. The third summation tells us that the space H_τ, which has the structure of $\mathrm{O}_m \times \mathrm{GL}_n$-module, is isomorphic to a tensor product $\tau \otimes \tilde{\tau}$ where $\tilde{\tau}$ is a GL_n-module determined by τ. The fourth summation further indicates that we may combine $\tilde{\tau} \cdot \mathbb{C}[r_{ij}^2]$ into a module V_τ for $\mathfrak{sp}_{\mathbb{C}}$. This module V_τ is irreducible and is determined by τ.

The modules V_τ will belong to the holomorphic discrete series if $m > 2n$, and will include all holomorphic discrete series when $m = 2n$ [Ge]. As long as $m \geq 2n$, the multiplications $\tilde{\tau} \cdot \mathbb{C}[r_{ij}^2]$ by $\mathbb{C}[r_{ij}^2]$ yield tensor product decompositions:

$$V_\tau = \tilde{\tau} \cdot \mathbb{C}[r_{ij}^2] \simeq \tilde{\tau} \otimes \mathbb{C}[r_{ij}^2]. \tag{6}$$

This enables us to prove Theorem 1 by establishing an appropriate result about the structure of the polynomial ring $\mathcal{P}(\mathbb{C}^{mn})$.

Let U_n denote the standard maximal unipotent subgroup of $\mathrm{GL}_n(\mathbb{C})$ of upper triangular matrices. The highest weight vectors for $K \subseteq \mathrm{Sp}(2n, \mathbb{R})$ acting on $\mathcal{P}(\mathbb{C}^{mn})$ are the invariant polynomials for U_n. As was remarked in the discussion leading up to the statement of Theorem 1, the algebra $\mathbb{C}[r_{ij}^2]^{U_n}$ of highest weight vectors in $\mathbb{C}[r_{ij}^2]$ is a polynomial algebra. We may describe its generators as follows. Consider the r_{ij}^2 to be the entries of a symmetric $k \times k$-matrix. Define δ_k to be the determinant of the leading $k \times k$ submatrix:

$$\delta_1 := r_{11}^2 \qquad \delta_2 := \begin{vmatrix} r_{11}^2 & r_{12}^2 \\ r_{12}^2 & r_{22}^2 \end{vmatrix} \qquad \delta_3 := \begin{vmatrix} r_{11}^2 & r_{12}^2 & r_{13}^2 \\ r_{12}^2 & r_{22}^2 & r_{23}^2 \\ r_{13}^2 & r_{23}^2 & r_{33}^2 \end{vmatrix}, \tag{7}$$

and so forth. Thus

$$\mathbb{C}[r_{ij}^2]^{U_n} = \mathcal{P}(\mathbb{C}^{mn})^{\mathrm{O}_m \times U_n} = \mathbb{C}[\delta_1, \delta_2, \ldots, \delta_n].$$

From the description of $\mathcal{P}(\mathbb{C}^{mn})$ given in formulas (5), we can see that Theorem 1 will follow from the following result.

Theorem 2. *Under the assumption that $m \geq 2n$, the algebra $\mathcal{P}(\mathbb{C}^{mn})^{U_n}$ of* $\mathrm{GL}_n(\mathbb{C})$ *highest weight vectors in $\mathcal{P}(\mathbb{C}^{mn})$ is a free module for the subalgebra $\mathbb{C}[r_{ij}^2]^{U_n} = \mathbb{C}[\delta_1, \delta_2, \ldots, \delta_n]$.*

This theorem can be regarded as a first step towards understanding the structure of $\mathcal{P}(\mathbb{C}^{mn})$ as an $\mathrm{O}_m \times \mathrm{GL}_n$-algebra. A final goal is to give an explicit description of the algebra of $\mathrm{O}_m \times \mathrm{GL}_n$ highest weight vectors $\mathcal{P}(\mathbb{C}^{mn})^{U \times U_n}$ where $U \subset \mathrm{O}_m$ is a maximal unipotent subgroup.

Example. In order to simplify the notation we will work with the orthogonal group O_m with respect to the quadratic form $\delta = \sum_1^m x_i x_{m-i+1}$. Then the lower triangular matrices in O_m form a maximal unipotent subgroup $U \subset \mathrm{O}_m$ and the diagonal matrices a maximal torus.

The description of $\mathcal{P}(\mathbb{C}^{mn})^{U \times U_n}$ in case $n = 1$ is easy:

$$\mathcal{P}(\mathbb{C}^m)^U \quad \mathbb{C}[x \quad \delta] \supset \mathcal{P}(\mathbb{C}^m)^{\mathrm{O}_m} \quad \mathbb{C}[\delta]$$

This is also expressed by the well-known decomposition formula of the space $\mathcal{P}_d(\mathbb{C}^m)$ of homogeneous functions of degree d as an O_m-module:

$$\mathcal{P}_d(\mathbb{C}^m) = \langle x_1^d \rangle_{\mathrm{O}_m} \oplus \delta \cdot \mathcal{P}_{d-2}(\mathbb{C}^m).$$

Here $\langle f \rangle_G$ denotes the G-module linearly spanned by the G-orbit of the element f. In particular, $\langle x_1^d \rangle_{\mathrm{O}_m}$ is the irreducible O_m-module of highest weight $d\omega_1$.

The case $n = 2$ is more complicated. We first introduce the following elements

$$\alpha_1 := x_{11} \quad \text{and} \quad \alpha_2 := \begin{vmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{vmatrix}.$$

Clearly, $\alpha_1^{a_1}\alpha_2^{a_2}$ is a $\mathrm{O}_m \times \mathrm{GL}_2$ highest weight vector of weight $a_1\omega_1 + a_2\omega_2$ (with respect to both groups) corresponding to the diagram (a_1+a_2, a_2). In fact, these are exactly the $\mathrm{O}_m \times \mathrm{GL}_2$ harmonic highest weight vectors, i.e. $\langle \alpha_1^{a_1}\alpha_2^{a_2} \rangle_{\mathrm{O}_m \times \mathrm{GL}_2} = H_\tau$ in the notation of formula (5) where τ is the irreducible O_m-module of highest weight $a_1\omega_1 + a_2\omega_2$. One can easily verify the following explicit Clebsch-Gordan decomposition in $\mathcal{P}(\mathbb{C}^{m2})$:

$$\begin{aligned} \langle \alpha_1 \rangle_{\mathrm{GL}_2} \cdot \langle \delta_1 \rangle_{\mathrm{GL}_2} &= \langle \alpha_1 \delta_1 \rangle_{\mathrm{GL}_2} \oplus \langle \beta_1 \rangle_{\mathrm{GL}_2} \\ \langle \alpha_1^2 \rangle_{\mathrm{GL}_2} \cdot \langle \delta_1 \rangle_{\mathrm{GL}_2} &= \langle \alpha_1^2 \delta_1 \rangle_{\mathrm{GL}_2} \oplus \langle \alpha_1 \beta_1 \rangle_{\mathrm{GL}_2} \oplus \langle \beta_2 \rangle_{\mathrm{GL}_2} \end{aligned}$$

where

$$\beta_1 := \begin{vmatrix} x_{11} & r_{11}^2 \\ x_{12} & r_{21}^2 \end{vmatrix} \quad \text{and} \quad \beta_2 := \begin{vmatrix} 0 & x_{11} & x_{12} \\ x_{11} & r_{11}^2 & r_{12}^2 \\ x_{12} & r_{21}^2 & r_{22}^2 \end{vmatrix}$$

are $\mathrm{O}_m \times \mathrm{GL}_2$ highest weight vectors of GL_2-weight $(2, 1)$ and $(2, 2)$, respectively. (δ_1 and δ_2 have been defined earlier.) Moreover, we have the following Lewis Carroll identity. (For the origin of the terminology, see [Do], [RR].)

$$\beta_2\delta_1 + \alpha_1^2\delta_2 + \beta_1^2 = 0.$$

Now one can show (see [ATZ]) that

$$\mathcal{P}(\mathbb{C}^{m2})^{U \times U_2} = \mathbb{C}[\alpha_1, \alpha_2, \beta_1, \beta_2, \delta_1, \delta_2]/(\beta_2\delta_1 + \alpha_1^2\delta_2 + \beta_1^2).$$

It follows that the monomials $\alpha_1^{a_1}\alpha_2^{a_2}\beta_1^{\varepsilon}\beta_2^{b_2}$ with $a_1, a_2, b_2 \geq 0$ and $\varepsilon = 0$ or 1, form a basis for $\mathcal{P}(\mathbb{C}^{m2})^{U \times U_2}$ as a $\mathbb{C}[\delta_1, \delta_2]$-module. The monomials with a fixed O_m-highest weight $\lambda = a_1\omega_1 + a_2\omega_2$ have the form $\alpha_1^{a_1-\varepsilon-2b_2}\alpha_2^{a_2}\beta_1^{\varepsilon}\beta_2^{b_2}$. From this description it is easy to explicitly verify the statements made so far.

From the above presentation, we can easily calculate the multiplicities for representations of K_2 (= the unitary group of 2×2 matrices) in holomorphic discrete series of $\mathrm{Sp}_4(\mathbb{R})$. As above, we indicate a dominant weight λ by a pair of integers (a_1, a_2), $a_1 \geq a_2$ where $\lambda = (a_1 - a_2)\omega_1 + a_2\omega_2$. Let $\lambda = (a_1, a_2)$ be the lowest highest weight of the holomorphic discrete series representation V_λ and let (b_1, b_2) denote a typical highest weight. Then the multiplicity of the representation of K_2 with highest weight (b_1, b_2) is described as follows. First, the multiplicity is zero unless the following conditions are satisfied:

$$b_1 \geq a_1, \quad b_2 \geq a_2, \quad \text{and} \quad b_1 + b_2 \equiv a_1 + a_2 \mod 2$$

We assume this from now on. Then

(i) $\operatorname{mult}_{(b_1,b_2)} V_\lambda = \min\{\frac{a_1-a_2+1}{2}, \lfloor\frac{b_1-a_1}{2}\rfloor + 1, \lfloor\frac{b_2-a_2}{2}\rfloor + 1, \frac{b_1-b_2+1}{2}\}$
if $a_1 - a_2$ is odd.
(Here $\lfloor x \rfloor$ denotes the greatest integer not exceeding x.)

(ii) $\operatorname{mult}_{(b_1,b_2)} V_\lambda = \min\{\frac{a_1-a_2}{2} + 1, \frac{b_1-a_1}{2} + 1, \frac{b_2-a_2}{2} + 1, \frac{b_1-b_2}{2} + 1\}$
if $a_1 - a_2$ and both $b_j - a_j$ are even.

(iii) $\operatorname{mult}_{(b_1,b_2)} V_\lambda = \min\{\frac{a_1-a_2}{2}, \frac{b_1-a_1+1}{2}, \frac{b_2-a_2+1}{2}, \frac{b_1-b_2}{2}\}$
if $a_1 - a_2$ is even and both $b_j - a_j$ are odd.

Remark. The phenomenon described in Theorem 2 seems to be quite special. First of all it is already very rare that the ring of U-invariants of a representation of a reductive group is a polynomial ring. (See [Br] where Brion classifies all irreducible representations of simple groups with this property. In our situation above the representation is $\wedge^2\mathbb{C}^n$.) Moreover, consider the following general question which illustrates the phenomenon. Let G be a reductive group with maximal unipotent subgroup U and let $\pi\colon X \to Y$ be an equivariant morphism of G-varieties such that $\mathcal{O}(X)$, the ring of regular functions on X, is a free module over $\mathcal{O}(Y)$. *Then when is* $\mathcal{O}(X)^U$ *a free module over* $\mathcal{O}(Y)^U$*?*

This certainly fails in general. For example let $G = \mathrm{SL}_2$, $X = Y \times \mathbb{C}^2$ and π the projection onto the first factor. If $\mathcal{O}(X)^U$ is free over $\mathcal{O}(Y)^U$ then $\mathcal{O}(X)^G \simeq \mathcal{O}(Y)^U$ is free over $\mathcal{O}(Y)^G$. Hence, the covariants for Y should be a free module over the invariants. This typically does not happen. A representation V with this property is called *cofree* and there are only finitely many for a given group G [Pop]. For simple groups G they have been classified by Schwarz [Schw], and the irreducible cofree representations V for semisimple groups by Littelmann [Li].

Proof of Theorem 2. The algebra $R := \mathcal{P}(\mathbb{C}^{mn})^{U_n}$ of U_n-invariants is finitely generated and Cohen-Macaulay ([KrP] section 18.7; cf. [Pop2]). Denote by X the corresponding affine variety $\operatorname{spec} R$, the maximal spectrum of R, and by $\delta\colon X \to \mathbb{C}^n$ the morphism $\delta(x) := (\delta_1(x), \delta_2(x), \ldots, \delta_n(x))$. Since $\mathbb{C}[\delta_1, \delta_2, \ldots, \delta_n]$ is the O_m-invariant ring R^{O_m} this means that the map δ is the quotient morphism by O_m ([Krf, II.3]). The essential step in the proof is to show that the zero fiber

$$\delta^{-1}(0) = \{x \in X \mid \delta_1(x) = \delta_2(x) = \cdots = \delta_n(x) = 0\}$$

has codimension n in X (Proposition 1 below). Since δ is homogeneous this implies that all fibers of δ have codimension n, i.e., δ is equidimensional. It follows that δ is flat because X is Cohen-Macaulay and $\mathbb{C}^n$ is smooth ([Mat, Theorem 23.1 and Corollary]). Decompose R into a direct sum of isotypic components with respect to O_m:

$$R = \bigoplus_{\tau} R^{\mathrm{O}_m,\tau}$$

Then each $R^{\mathrm{O}_m,\tau}$ is a finitely generated graded module over R^{O_m} ([Krf, II.3.2]). Because of flatness, every summand $R^{\mathrm{O}_m,\tau}$ is therefore a free R^{O_m}-module, and Theorem 2 follows. □

As in the proof above we denote by X the affine variety $\operatorname{spec} R$, the maximal spectrum of the U_n-invariant functions $R = \mathcal{P}(\mathbb{C}^{mn})^{U_n}$ on $\mathbb{C}^{mn}$. Let us call a sequence $f_1, f_2, \ldots, f_r$ of regular functions on an irreducible variety Z *good* if the

Proposition 1. *Under the assumption $m \geq 2n$ the functions $\delta_1, \delta_2, \dots, \delta_n$ form a good sequence on X.*

Proof. By classical invariant theory the ring R of U_n-invariant functions on $\mathbb{C}^{mn}$, the so-called *primary covariants*, is generated by the $k \times k$-minors extracted from the first k columns of $\mathbb{C}^{mn}$ for $k = 1, 2, \dots, n$. (This follows immediately from the well-known fact that the $U_m \times U_n$-invariants are generated by the principal minors, see [Krf, III.3.7] or [Ho]). Thus we obtain the following description of the variety X as subvariety of

$$V \times \bigwedge^2 V \times \bigwedge^3 V \times \cdots \times \bigwedge^n V$$

where $V := \mathbb{C}^m$. Let $w := (e_1, e_1 \wedge e_2, e_1 \wedge e_2 \wedge e_3, \dots, e_1 \wedge \cdots \wedge e_n) \in W$. Then

$$X = \overline{\mathrm{GL}_m \cdot w},$$

the closure of the GL_m orbit of the point w because $\mathrm{GL}_m \times U_n$ has a dense orbit in $\mathbb{C}^{mn}$. In fact, even $B_m \times U_n$ has a dense orbit where B_m are the upper triangular matrices, and thus X is multiplicity-free and contains only finitely many GL_m-orbits. We can describe them in the following way. For any subset $J \subset N := \{1, 2, \dots, n\}$, $J = \{j_1 < j_2 < \cdots < j_s\}$ we define $w_J \in W$ by

$$(w_J)_j := \begin{cases} 0 & \text{if } j \notin J \\ e_1 \wedge e_2 \wedge \cdots \wedge e_j & \text{if } j \in J \end{cases}$$

Then

$$X = \overline{\mathrm{GL}_m \cdot w} = \bigcup_{J \subset N} \mathrm{GL}_m \cdot w_J$$

Let P_J be the parabolic subgroup stabilizing the flag $\mathbb{C}^{j_1} \subset \mathbb{C}^{j_2} \subset \cdots \subset \mathbb{C}^{j_s} \subset \mathbb{C}^m$. Then the stabilizer of w_J is the subgroup Q_J of P_J subject to the condition that the automorphism induced on $\mathbb{C}^{j_k}/\mathbb{C}^{j_{k-1}}$ has determinant 1 for $k = 1, \dots, s$.

Next we remark that the functions δ_k are defined on all of W:

$$\delta_k(\omega_1, \omega_2, \dots, \omega_n) = (\omega_k, \omega_k)$$

where $(\quad,\quad)$ is the symmetric bilinear form on $\bigwedge^k V$ induced by the standard form on V, i.e., the k-fold wedge products from an orthonormal basis for V form an orthonormal basis for $\Lambda^k V$. If $\omega_k = v_1 \wedge \cdots \wedge v_k$ then $(\omega_k, \omega_k) = 0$ if and only if the subspace of V spanned by $v_1, \dots, v_k$ is degenerate with respect to the standard form on V. In particular we see that δ_j vanishes on the orbit $\mathrm{GL}_m \cdot w_J$ if $j \notin J$. The following Main Lemma implies that the locus of common zeroes of $\delta_1, \dots, \delta_n$ has codimension $\#J$ in the orbit $\mathrm{GL}_m \cdot w_J$. Now we remark that $\dim Q_J - \dim Q_N \geq n - \#J$ which means that the orbit $\mathrm{GL}_m \cdot w_J$ has codimension at least $n - \#J$ in X. This proves Proposition 1. $\square$

Main Lemma. *The functions $\{\delta_j \mid j \in J\}$ form a good sequence on the orbit $\mathrm{GL}_m \cdot w_J$.*

Proof. Consider the orbit map $p_J \colon \mathrm{GL}_m \to \mathrm{GL}_m \cdot w_J$. For any $j \in J$ the pull-back $\tilde{\delta}_j = p_J \circ \delta_j$ has the following interpretation: For any matrix $g \in \mathrm{GL}_m$ with column vectors $v_1, v_2, \dots, v_m$ we have $\tilde{\delta}_j(g) = 0$ if and only if the span of $v_1, \dots, v_j$ in

$V(=\mathbb{C}^m)$ is degenerate with respect to the form $(\ ,\ )$. If the sequence $(\tilde{\delta_j} \mid j \in J)$ is good then its zero locus $Z_J \subset \mathrm{GL}_m$ has codimension $\#J$ in GL_m. Moreover, Z_J is stable under right multiplication by Q_J. It follows that the image $\overline{Z_J} := p_J(Z_J)$ is the zero set of $\{\delta_j \mid j \in J\}$ and has codimension $\#J$ in the orbit $\mathrm{GL}_m \cdot w_J$. Thus it remains to prove that the sequence $\tilde{\delta_1}, \tilde{\delta_2}, \dots, \tilde{\delta_n}$ of functions on GL_m, and, more generally, that every subsequence of the $\tilde{\delta_j}$ is good.

For this purpose consider the map $\tilde{\delta} := (\tilde{\delta_1}, \tilde{\delta_2}, \dots, \tilde{\delta_n})\colon \mathrm{GL}_m \to \mathbb{C}^n$. By definition, it can be decomposed in the form

$$\mathrm{GL}_m \xrightarrow{p} \mathrm{Sym}_m \xrightarrow{\mu} \mathbb{C}^n, \qquad g \overset{p}{\mapsto} g^t g = A \overset{\mu}{\mapsto} (\mu_1(A), \mu_2(A), \dots, \mu_n(A))$$

where Sym_m are the symmetric $m \times m$-matrices and $\mu_1(A), \dots, \mu_n(A)$ are the first n principal minors of A. The first map is a principal O_m-bundle over its image and therefore equidimensional (i.e., the inverse image under p of a closed subset of codimension d has codimension d). Thus, it remains to show that $\mu_1, \mu_2, \dots, \mu_n$ form a good sequence on Sym_m, i.e., that the zero fiber $\mu^{-1}(0)$ has codimension n in Sym_m. In fact, since the functions μ_i are all homogeneous this implies that $\mu_1, \mu_2, \dots, \mu_n$ form a *regular sequence* and so every subsequence $(\mu_j \mid j \in J)$ is good, too (cf. [Mat, Theorem 17.4]). It follows that the locus of common zeroes in Sym_m has codimension $\#J$ and hence the same holds for the locus of common zeroes of $(\tilde{\delta_j} \mid j \in J)$ in GL_m. So everything is reduced to the following proposition.
□

Proposition 2. *The morphism* $\mu\colon \mathrm{Sym}_m \to \mathbb{C}^m$, $A \mapsto (\mu_1(A), \mu_2(A), \dots, \mu_m(A))$, *where* $\mu_i(A)$ *is the ith principal minor of* A, *is equidimensional.*

The following proof was communicated to us by Gerry Schwarz. It replaces our more complicated argument using induction on m.

Proof. It is enough to find a regular sequence for $\mathcal{O}(\mathrm{Sym}_m)$ which contains the μ_i. So, first choose linear functions $l_1, \dots, l_r$, $r := \dim \mathrm{Sym}_m - m = \binom{m}{2}$, such that their zeroes consist of matrices of the form:

$$A = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_m \\ a_2 & a_3 & \dots & a_m & 0 \\ a_3 & \vdots & & 0 & 0 \\ \vdots & a_m & 0 & 0 & \vdots \\ a_m & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Then the functions μ_i send the matrix A above to

$$\mu_1(A) = a_1,\ \mu_2(A) = a_1 a_3 - a_2^2,\ \mu_3(A) = -a_3^3 - a_2^2 a_5 - a_1 a_4^2 + a_1 a_3 a_5 + 2 a_2 a_3 a_4,$$

etc.. We see that $\mu_i(A)$ contains the variable a_i to the pure power i, and the rest of the expression consists of monomials involving at least one of the previous variables $a_1, a_2, \dots, a_{i-1}$. Thus the $\binom{m+1}{2}$ equations

$$l_1(A) = l_2(A) = \dots = l_r(A) = \mu_1(A) = \mu_2(A) = \dots = \mu_m(A) = 0$$

force $A = 0$. It follows that the l_j followed by the μ_i form a homogeneous system of parameters in the polynomial ring $\mathcal{O}(\mathrm{Sym}_m)$, hence a regular sequence (cf. [Mat, Theorem 17.4]). □

Appendix

The essential point in the proof is the Main Lemma where we show that the functions $\{\delta_j \mid j \in J\}$ form a good sequence on the orbit $\mathrm{GL}_m \cdot w_J \simeq \mathrm{GL}_m / Q_J$ (see the proof of Proposition 1). In this appendix we make a more careful analysis of the situation and give a very precise description of the zero locus of the collection $\{\delta_b\}_{1\leq b\leq \ell}$. The discussion could easily be extended to describe the zero locus of any collection of the δ_b.

We can regard functions on any one of the orbits GL_m / Q_J as functions on GL_m which happen to be invariant under right translations by elements of Q_J. Since the δ_b are invariant under left translation by the orthogonal groups O_m, their zero sets will evidently also be invariant under left translation by O_m. Further, the fact that the δ_b are also homogeneous means that they are, not invariant, but eigenfunctions under right translations by the full parabolic P_J containing Q_J. Thus, although the δ_b are not P_J invariant, their zero sets *will* be P_J invariant. In sum, then, when we ask about the zero set of the functions δ_b on the variety GL_m / Q_J, we are talking about a union of (O_m, P_J) double cosets, that is, a subset of the double coset space $\mathrm{O}_m \setminus \mathrm{GL}_m / P_J$.

If $B \subset P_J$ is a Borel subgroup, then the (O_m, P_J) double cosets are unions of (O_m, B) double cosets. It turns out that there are only a finite number of these; and there is a strong analogy between the description of $\mathrm{O}_m \setminus \mathrm{GL}_m / B$ and the Bruhat decomposition which describes $B\setminus \mathrm{GL}_m / B$. (The finiteness of these double coset spaces illustrates a result of Brion and Vinberg (cf. [Br2], [Vi]) which asserts the finiteness of the B-orbits in any multiplicity-free variety.)

The detailed description of $H\setminus G/B$ where G is a semisimple group, B a Borel subgroup and H a symmetric subgroup has been developed in [Ma], [Sp]. We will describe the specialization of their general results to the case at hand. We may think of (O_m, B) double cosets as GL_m-orbits of pairs $(\beta, \mathcal{F})$, where β is an inner product on $\mathbb{C}^m$, and $\mathcal{F} = \{\{0\} \subset V_1 \subset V_2 \subset V_3 \subset \cdots \subset V_m = \mathbb{C}^m\}$ is a complete flag in $\mathbb{C}^m$. Here GL_m acts on inner products and on flags in the usual ways, and acts on pairs by the product action. Somewhat more precisely, we have a diagram

$$\begin{tikzcd} \mathrm{GL}_m \times \mathrm{GL}_m \arrow[r] \arrow[d] & \mathrm{GL}_m / \mathrm{O}_m \times \mathrm{GL}_m / B \arrow[d] \\ \mathrm{GL}_m \arrow[r] & \mathrm{O}_m \setminus \mathrm{GL}_m / B \end{tikzcd} \tag{A1}$$

where the left vertical arrow is the principal GL_m-fibration $(g_1, g_2) \mapsto g_1^{-1} g_2$, and the upper horizontal arrow is the quotient mapping which is a principal $\mathrm{O}_m \times B$-fibration. It follows that the codimension of a GL_m-orbit in $\mathrm{GL}_m / \mathrm{O}_m \times \mathrm{GL}_m / B$ is equal to the codimension of a (O_m, B) double coset to which it is mapped by the right hand side vertical arrow.

Claim A2. *Given such a pair $(\beta, \mathcal{F})$, we can find a basis $\{v_j\}_{1\leq j\leq m}$ for $\mathbb{C}^m$ such that*

(i) *$\{v_j\}_{1\leq j\leq k}$ is a basis for V_k, for each $k, 1 \leq k \leq m$.*

(ii) *There is a permutation $\sigma(\beta, \mathcal{F}) = \sigma$ of $\{1, 2, 3, \ldots, m\}$, of order 2, such that $\beta(v_i, v_j) = \delta_{i\sigma(j)} = \delta_{\sigma(i)j}$.*

Indeed, consider V_1. If V_1 is not isotropic for the inner product β, we can find multiplication by ±1.) Then let $Y = V_1^{\perp}$, where the orthogonal complement is taken with respect to β. Define a complete flag $\mathcal{F}' = \{V_j'\}_{1\leq j\leq m-1}$ in Y by setting

$$V_j' = V_{j+1}' \cap Y.$$

By induction on dimension, we can assume the claim is true for the pair $(\beta\mid_Y, \mathcal{F}')$. Let $\{v_j'\}$ be the basis of Y as specified in the claim. Then setting $v_j = v_{j-1}'$ for $j \geq 2$, we see that $\{v_j\}_{1\leq j\leq m}$ will satisfy the claim for $(\beta, \mathcal{F})$.

Consider, on the other hand, the possibility that V_1 is isotropic for β. Let a be the smallest index such that $V_a \not\subset V_1^{\perp}$. Choose v_1 arbitrarily in V_1, and choose v_a in V_a such that $\beta(v_1, v_a) = 1$, and $\beta(v_a, v_a) = 0$. Note that, if we have u in V_a such that $\beta(v_1, u) = 1$, then we can set $v_a = u - (\beta(u,u)/2)v_1$. Thus we see that, if we have a satisfactory v_a, we can add to it any element in V_{a-1}, then further add an appropriate multiple of v_1, to obtain another suitable candidate for v_a. Together with the fact that v_1 is only determined up to multiples, this lets us see that there is an $(a-1)$-dimensional family of satisfactory pairs $\{v_1, v_a\}$.

Let P be the plane spanned by v_1 and v_a, and set $Y = P^{\perp}$. Define a complete flag $\mathcal{F}' = \{V_j'\}_{1\leq j\leq m-2}$ in Y by the recipe

$$\begin{aligned} V_j' &= V_{j+1} \cap Y \qquad \text{for } 1 \leq j \leq a-2;\\ V_j' &= V_{j+2} \cap Y \qquad \text{for } a-1 \leq j \leq m-2. \end{aligned}$$

By induction on dimension we may assume we have a basis $\{v_j'\}$ for Y, which basis satisfies the claim for the pair $(\beta\mid_Y, \mathcal{F}')$. Now define

$$\begin{aligned} v_j &= v_{j-1}' \qquad \text{for } 2 \leq j \leq a-1,\\ v_j &= v_{j-2}' \qquad \text{for } a+1 \leq j \leq m. \end{aligned}$$

It is easy to check that the basis $\{v_j\}$ satisfies the claim for the pair $(\beta, \mathcal{F})$. □

The above procedure also allows one to compute $\dim(B_{\mathcal{F}} \cap O_\beta)$, where $B_{\mathcal{F}}$ is the stabilizer of $\mathcal{F}$ in GL_m, and O_β is the isometry group of β. This will also be the codimension of GL_m-orbit of $(B, \mathcal{F})$, since $\dim B_V + \dim O_\beta = \dim \mathrm{GL}_m = m^2$. As mentioned above, this is also the codimension of the corresponding double coset. Since we had an $a-1$ dimensional set from which to choose the pair (v_1, v_a), we have the relation

$$\dim(B_{\mathcal{F}} \cap O_\beta) = a - 1 + \dim(B_{\mathcal{F}'} \cap (O_\beta \cap \mathrm{GL}(Y'))). \tag{A3}$$

Note that this is valid also in the case when $a = 1$.

We observe that $a-1$ is the distance between 1 and $a = \sigma(1)$. This gives us the main term in the expression for $\dim(B_{\mathcal{F}} \cap O_\beta)$. For a pair $(c < d)$ of integers, let $\sigma(c,d)$ denote the transposition which exchanges c and d, and leaves all other integers fixed. Given pairs $(c < d)$ and $(c' < d')$ of integers, we say that they are *linked* if $c < c' < d < d'$, or if $c' < c < d' < d$. We say the permutations $\sigma(c,d)$ and $\sigma(c',d')$ are linked if the pairs (c,d) and (c',d') are.

Given an element σ of order two in the symmetric group on m letters, we can factor it into a product of transpositions:

$$\sigma = \prod_i \sigma(c_i, d_i). \tag{A4}$$

Lemma A5. *If the involution σ attached to the pair $(\beta, \mathcal{F})$ is factored as in formula* (A4), *then*

$$\dim(B_{\mathcal{F}} \cap O_\beta) = \sum_i (d_i - c_i) - \#\{(i,j) \mid \sigma(c_i, d_i) \text{ and } \sigma(c_j, d_j) \text{ are linked}\}.$$

Remark. We note that if we multiply σ by another factor $\sigma(c, d)$, the right hand side of this formula increases, since the number of pairs (c_i, d_i) with which (c, d) can be linked is at most $c - d - 1$.

Proof. If we assume the formula is true for Y, then it remains true for $\mathbb{C}^m$, by formula (A3), since if (c_i, d_i) is linked to $(1, a)$, it becomes $(c_i - 1, d_i - 2)$ in Y, while if it is not linked, it becomes $(c_i - e, d_i - e)$, where $e = 1$ or 2, as the case may be. ☐

Proof of the Main Lemma. Now let β_0 denote our fixed standard inner product on $\mathbb{C}^m$, and let δ_j be the O_m-invariant primary covariants of formula (7). Consider what (O_m, B) double cosets can be in the zero-locus of $\{\delta_j\}_{1 \le j \le \ell}$. For the flag $\mathcal{F}$ to be in the zero locus of δ_j, the space V_j should be singular for β_0. This follows by combining the diagram (A1) with the description of the zero locus of δ_j as given in the proof of Proposition 1.

This means, for the basis $\{v_j\}$ adapted to the pair $(\beta_0, \mathcal{F})$ as per Claim A2, that not every v_c with $c \le j$ is paired with a v_d with $d \le j$. In other words, some v_c with $c \le j$ is paired with v_d with $d > j$. Or, the set $\{1, 2, 3, \ldots, j\}$ is not invariant under the involution $\sigma(\beta_0, \mathcal{F})$. That is, there is some factor $\sigma(c_i, d_i)$ of σ, with $c_i \le j < d_i$.

Suppose this holds for all j up to $j = \ell$. Taking $j = 1$, we see that 1 must not be fixed by σ. Hence in the factorization (A4) of σ, there is a factor $\sigma(1, d_1)$. If $d_1 > \ell$, then $\mathcal{F}$ is guaranteed to be in the zero locus of $\{\delta_j\}_{1 \le j \le \ell}$. However, if $d_1 \le \ell$, there must be another factor $\sigma(c_2, d_2)$ of σ, with $c_2 < d_1 < d_2$. If there is more than one such, we may chose the one with d_2 as large as possible. If $d_2 > \ell$, then we now are assured that $\mathcal{F}$ is in the zero locus of $\{\delta_j\}_{1 \le j \le \ell}$. If however, we still have $d_2 \le \ell$, then there must be a third factor $\sigma(c_3, d_3)$ of σ, with $c_3 < d_2 < d_3$. Since we choose d_2 as large as possible at the previous stage, we know that $c_3 > d_1$. We may now select d_3 to be as large as possible. Continuing in this fashion, we conclude that σ has a set of factors $\sigma(c_i, d_i)$ for $1 \le i \le r$, such that

$$1 = c_1 < c_2 < d_1 < c_3 < d_2 < c_4 < d_3 < \cdots < c_r < d_{r-1} \le \ell < d_r.$$

From Lemma A5 and the Remark following it, we see that the codimension of the coset containing $\mathcal{F}$ is at least equal to

$$\sum_{i=1}^{r} (d_i - c_i) - (r - 1) \ge (d_1 - 1) + \sum_{i=2}^{r} (d_i - d_{i-1} + 1) - (r - 1) \ge d_r - 1 \ge \ell,$$

as desired.

Remark. Using the above inequalities, we can be precise about which cosets have codimension exactly ℓ. They would be the ones for which $c_i = d_{i-1} - 1$, and $d_r = \ell + 1$. Thus they correspond to all possible sequences where $d_i - d_{i-1} \geq 2$ if $d_i \leq \ell$ (to allow for the insertion of c_{i+1}). The number of such sequences is the ℓ-th Fibonacci number.

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Department of Mathematics, Yale University,
New Haven, CN 06520-8283, USA
E-mail address: `howe-roger@math.yale.edu`

Mathematisches Institut, Universität Basel
Rheinsprung 21, CH-4051 BASEL, Switzerland
E-mail address: `kraft@math.unibas.ch`