

# FREE $\mathbb{C}^+$ -ACTIONS ON AFFINE THREEFOLDS

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ABSTRACT. We study algebraic actions of the additive group  $\mathbb{C}^+$  on an affine threefold  $X$  and prove a smoothness property for the quotient morphism  $\pi: X \rightarrow X//\mathbb{C}^+$ . Then, following SHULIM KALIMAN, we give a proof of the conjecture that every free  $\mathbb{C}^+$ -action on affine 3-space  $\mathbb{C}^3$  is a translation.

## §1. THE MAIN RESULT

These notes are about algebraic actions of the additive group  $\mathbb{C}^+$  on complex affine threefolds and, in particular, about the following conjecture which was recently proved by SHULIM KALIMAN [Ka04].

**Conjecture.** *Every free  $\mathbb{C}^+$ -action on  $\mathbb{C}^3$  is a translation, i.e. there is a closed subset  $S \subset \mathbb{C}^3$  such that the natural morphism  $\mathbb{C}^+ \times S \xrightarrow{\sim} \mathbb{C}^3$ ,  $(s, x) \mapsto s \cdot x$ , is an isomorphism.*

In fact, KALIMAN proves a more general statement. Together with some results from the joint paper [KaS04] with SAVELIEV one has now the following general theorem.

**Theorem (KALIMAN-SAVELIEV).** *Every free  $\mathbb{C}^+$ -action on a smooth contractible affine threefold  $X$  is a translation.*

For more information about this type of problems and additional background we refer to the survey [Kr96]. In particular, the corresponding result in dimension 2 goes back to RENTSCHLER [Re68] who classified the  $\mathbb{C}^+$ -actions on  $\mathbb{C}^2$ , and the conjecture is wrong in dimension 4 as shown by WINKELMANN in [Wi90].

The text is organized as follows. In the next two sections we describe an approach to the problem in a more general setting and work out the algebraic “obstructions” for actions to be locally trivial. This part is partially based on discussions I had with DAVE FINSTON and JAMES DEVENEY some time ago where we developed a strategy to prove the conjecture using the smoothness of the quotient morphism  $\pi: \mathbb{C}^3 \rightarrow \mathbb{C}^3//\mathbb{C}^+$ . Looking again at their paper [DeF00] I realized that the smoothness of  $\pi$  can be proved under more general assumptions (see Proposition 1 in §2).

In the last section, using this smoothness property of the quotient map and some topological considerations due to KALIMAN we are able to give a short proof of the

Theorem above, where we assume, in addition, that the quotient  $X//\mathbb{C}^+$  is smooth (see the Main Theorem in §4).

It should be pointed out that most results here generalize to free actions of a unipotent group  $U$  of dimension  $n$  on an  $(n+2)$ -dimensional affine variety with the appropriate properties.

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## §2. THE BASIC SETUP

Our base field is the field  $\mathbb{C}$  of complex numbers. Let  $X$  be an affine variety of dimension 3 with a non-trivial action of the additive group  $\mathbb{C}^+$ . Assume that  $X$  is *factorial*, i.e. that the coordinate ring  $\mathcal{O}(X)$  is a unique factorization domain. Denote by  $X//\mathbb{C}^+$  the *algebraic quotient*. This means that  $X//\mathbb{C}^+$  is the affine variety with coordinate ring  $\mathcal{O}(X)^{\mathbb{C}^+}$ , the *invariant ring* under  $\mathbb{C}^+$ , which is finitely generated by a well-known result of ZARISKI [Za54]. Moreover,  $\pi = \pi_X: X \rightarrow X//\mathbb{C}^+$  denotes the *quotient morphism* defined by the inclusion  $\mathcal{O}(X)^{\mathbb{C}^+} \subset \mathcal{O}(X)$ .

**Lemma 1.** *The quotient  $Y := X//\mathbb{C}^+$  is a normal and factorial surface. If  $C \subset Y$  is an irreducible curve, then the inverse image  $\pi^{-1}(C)$  is reduced and irreducible. Moreover,  $\pi: X \rightarrow Y$  is equidimensional, i.e. every non-empty fiber has dimension 1, and  $Y \setminus \pi(X)$  is finite.*

*Proof.* The first part is well-known. It essentially uses the fact that the group  $\mathbb{C}^+$  has no characters.

An irreducible closed curve  $C \subset Y$  is defined by an irreducible function  $f \in \mathcal{O}(Y)$  which remains irreducible when considered as an element in  $\mathcal{O}(X)$ . Thus the schematic inverse image of  $C$  is reduced and irreducible.

Assume that a fiber  $\pi^{-1}(y)$  has an irreducible component  $Z$  of dimension 2. Then  $Z$  is defined by an irreducible function  $f \in \mathcal{O}(X)$ . Since  $Z$  is stable under  $\mathbb{C}^+$  the function  $f$  is invariant. If  $h$  is any other invariant, then  $h$  is constant on  $Z$ ,  $h|_Z \equiv c$ , and so  $h - c$  is a multiple of  $f$ . But this implies that the invariant ring is  $\mathbb{C}[f]$ , contradicting the fact that  $\dim Y = 2$ .

Finally, if  $Y \setminus \pi(X)$  is not finite, then  $C := \overline{Y \setminus \pi(X)}$  is a curve and so  $F := \pi^{-1}(C)$  is of dimension 2. This implies that  $\pi(F)$  is of dimension 1 which contradicts the fact that  $\pi(F)$  is contained in the finite subset  $C \cap \pi(X)$  of  $Y$ .  $\square$

**Proposition 1.** *Let  $x \in X \setminus X^{\mathbb{C}^+}$  and assume that  $X$  is smooth in  $x$  and  $Y$  is smooth in  $\pi(x)$ . Then  $\pi$  is smooth in  $x$ .*

*Proof.* (a) We first claim that there are only finitely many  $y \in Y \setminus \pi(X^{\mathbb{C}^+})$  such that the fiber  $\pi^{-1}(y)$  is not reduced. Otherwise there is an irreducible curve  $C \not\subset \pi(X^{\mathbb{C}^+})$  such that  $\pi^{-1}(y)$  is not reduced for almost all  $y \in C$ . Since  $F := \pi^{-1}(C)$  is reduced and irreducible by Lemma 1 this implies that almost all fibers of the morphism  $\pi|_F: F \rightarrow C$  are non-reduced. On the other hand the morphism  $\pi|_F$  is smooth on a dense open set which is stable under  $\mathbb{C}^+$ . It follows that almost all (schematic) fibers of  $\pi|_F$  are smooth, hence reduced, and we end up with a contradiction.

(b) Now let  $x_0 \in X \setminus X^{\mathbb{C}^+}$  such that  $X$  is smooth in  $x_0$  and  $Y$  is smooth in  $y_0 := \pi(x_0)$ , and assume that  $\pi$  is not smooth in  $x_0$ . We will show that this leads to a contradiction. We can find an irreducible closed surface  $S \subset X$  containing  $x_0$  and smooth in  $x_0$  such that  $T_{x_0}X = T_{x_0}S \oplus T_{x_0}(\mathbb{C}^+x_0)$ , i.e.  $S$  is transversal to the  $\mathbb{C}^+$ -orbit through  $x_0$ . It follows that there is an open dense subset  $S' \subset S$  containing  $x_0$  such that the following holds: (i)  $S'$  is smooth and  $\pi(S') \subset Y$  is open and smooth; (ii)  $S'$  is transversal to the orbits  $\mathbb{C}^+x$  for all  $x \in S'$ . In particular,  $S' \cap X^{\mathbb{C}^+} = \emptyset$ .

It follows that  $\pi|_{S'}$  is smooth in  $x \in S'$  if and only if  $\pi$  is smooth in  $x$ . It is well-known that the set of points of  $S'$  where  $\pi|_{S'}$  is not smooth is either empty or a curve, hence does not contain isolated points. By construction,  $\pi|_{S'}$  is not smooth in  $x_0$  and, by (a), the point  $x_0 \in S'$  is an isolated point with this property. Thus we end up with a contradiction which finally proves the proposition.  $\square$

**Corollary 1.** *If  $X$  and  $Y$  are both smooth and the  $\mathbb{C}^+$ -action fixed point free then the quotient morphism  $\pi: X \rightarrow Y$  is smooth.*

*Remark 1.* In case  $X = \mathbb{C}^3$  this is proved in [DeF00] by a direct argument. In this situation the quotient is  $\mathbb{C}^2$  by a result of MIYANISHI [Mi80].

*Remark 2.* It is shown in [KaS04] that for any smooth contractible affine threefold  $X$  the quotient  $X//\mathbb{C}^+$  is smooth. Thus the quotient map is smooth in this case. (Here we use the fact that a contractible affine smooth threefold is factorial.)

It is helpful to compare the results of this and the subsequent sections with the following example of a non-free  $\mathbb{C}^+$ -action on  $\mathbb{C}^3$ .

**Example.** Consider the  $\mathbb{C}^+$ -action on  $\mathbb{C}^3$  given by the nilpotent derivation  $\delta := x\frac{\partial}{\partial y} - 2y\frac{\partial}{\partial z}$ . Then the quotient  $\mathbb{C}^3//\mathbb{C}^+$  is isomorphic to  $\mathbb{C}^2$  and the quotient morphism is given by

$$\pi: \mathbb{C}^3 \rightarrow \mathbb{C}^2, \quad (x, y, z) \mapsto (x, xz + y^2).$$

The fixed point set  $(\mathbb{C}^3)^{\mathbb{C}^+}$  is the  $z$ -axis  $\{x = y = 0\} \subset \mathbb{C}^3$ . It equals  $\pi^{-1}(0)$  and coincides with the points where  $\pi$  is not smooth. The fibers  $\pi^{-1}((0, v))$  for  $v \neq 0$  consist of two orbits, and  $\pi: \mathbb{C}^3 \setminus \{x = 0\} \rightarrow \mathbb{C}^2 \setminus \{u = 0\}$  is a trivial  $\mathbb{C}^+$ -bundle.

### §3. CONFIGURATION OF “BAD” CURVES

From now on we make the following assumptions:

*$X$  is an affine, factorial and smooth variety of dimension 3 with a fixed point free  $\mathbb{C}^+$ -action such that the quotient  $Y := X//\mathbb{C}^+$  is smooth.*

It then follows from Proposition 1 that the quotient morphism  $\pi: X \rightarrow Y$  is smooth. Consider the following subset  $E \subset Y$ :

$$E := \{y \in Y \mid \pi^{-1}(y) \text{ is not an orbit}\}.$$

Observe that  $E$  contains the finite set  $Y \setminus \pi(X)$ .

**Proposition 2.** *If  $E$  is finite, then  $X$  is a principal  $\mathbb{C}^+$ -bundle over  $\pi(X)$ . In particular,  $E = Y \setminus \pi(X)$ .*

*Proof.* Consider the fiber product

$$\begin{array}{ccc} Z & \xrightarrow{\mu} & X \\ \downarrow \pi_Z & & \downarrow \pi \\ X & \xrightarrow{\pi} & Y \end{array}$$

where the horizontal  $\pi$  is considered as (a smooth) base change and so the  $\mathbb{C}^+$ -action on  $Z$  is such that  $\mu$  is equivariant. In addition,  $\pi_Z$  is the quotient morphism since almost all fibers are orbits, by assumption (use [Kr85, II.3.4 Lemma and Bemerkung]). Clearly,  $\pi_Z$  (and hence  $Z$ ) is smooth. Moreover, there is an equivariant morphism

$$\varphi: \mathbb{C}^+ \times X \rightarrow Z$$

such that  $\mu \circ \varphi_Z = \text{pr}_X$ , the projection onto  $X$ , and  $\mu \circ \varphi: \mathbb{C}^+ \times X \rightarrow X$  is given by the  $\mathbb{C}^+$ -action on  $X$ . Define  $\tilde{E} := \pi^{-1}(E) \subset X$ . Since, by construction, every fiber of  $\pi$  over  $Y \setminus E$  is an orbit, the same holds for the fibers of  $\pi_Z$  over  $X \setminus \tilde{E}$ , and so  $\varphi$  induces a bijective morphism

$$\varphi': \mathbb{C}^+ \times X \setminus \tilde{E} \xrightarrow{\sim} Z \setminus \pi_Z^{-1}(\tilde{E}).$$

Since  $E$  is finite we see that  $\text{codim}_X \tilde{E} \geq 2$  and so  $\text{codim}_Z \pi_Z^{-1}(\tilde{E}) \geq 2$ . Hence  $\varphi$  is an isomorphism (see [Kr85, II.3.4 Lemma and Bemerkung]). Thus the morphism  $\pi: X \rightarrow \pi(X)$  becomes a trivial  $\mathbb{C}^+$ -bundle after a smooth surjective base change, and the claim follows.  $\square$

For a (complex) variety  $Z$  we denote by  $\chi(Z)$  its *Euler characteristic* with respect to the usual  $\mathbb{C}$ -topology. We refer to [KrP85, Appendix] for some basic properties of  $\chi$  with respect to complex algebraic varieties and for more details.

**Corollary 2.** *If  $E$  is finite then the following statements are equivalent:*

- (i)  $\chi(X) = \chi(Y)$ ;
- (ii)  $\pi$  is surjective;
- (iii)  $\pi: X \rightarrow Y$  is a trivial  $\mathbb{C}^+$ -bundle;
- (iv) The  $\mathbb{C}^+$ -action is a translation, i.e.  $X \simeq \mathbb{C}^+ \times S$  for some closed  $S \subset X$ .

*Proof.* Since  $X$  is a locally trivial  $\mathbb{C}^+$ -bundle over  $\pi(X)$  we have  $\chi(X) = \chi(\pi(X))$ . Moreover,  $\chi(\pi(X)) = \chi(Y) - d$  where  $d$  is the cardinality of  $Y \setminus \pi(X)$  (see [KrP85, Appendix]). This proves the equivalence of (i) and (ii).

The assertion (iii) follows from (ii) since every principal  $\mathbb{C}^+$ -bundle over an affine variety is trivial, and (iii) obviously implies (iv). Finally, if  $X \simeq \mathbb{C}^+ \times S$  then the quotient morphism  $\pi: X \rightarrow Y$  corresponds to the projection  $\mathbb{C}^+ \times S \rightarrow S$ , and so  $\pi$  is surjective. Hence (ii).  $\square$

The second part of Proposition 2 generalizes to the following results which states that the only possible isolated points of  $\overline{E}$  are in  $Y \setminus \pi(X)$ .

**Corollary 3.** *If the set  $E$  is not finite then  $\overline{E \cap \pi(X)}$  is equidimensional of dimension 1 and the induced morphism*

$$X' := X \setminus \pi^{-1}(\overline{E}) \rightarrow \pi(X')$$

*is a principal  $\mathbb{C}^+$ -bundle.*

*Proof.* Let  $C \subset \overline{E \cap \pi(X)}$  be the union of the 1-dimensional components. Then  $C$  is defined by an invariant function  $f \in \mathcal{O}(Y)$  and  $X_f = X \setminus \pi^{-1}(C)$  is a  $\mathbb{C}^+$ -stable affine open subvariety of  $X$ . Moreover,  $\pi|_{X_f}: X_f \rightarrow Y_f$  is the quotient morphism (see Remark 3 below). Since  $E \cap Y_f$  is finite we can apply Proposition 2 and conclude that  $\pi|_{X_f}: X_f \rightarrow \pi(X_f)$  is a principal  $\mathbb{C}^+$ -bundle. Thus  $\pi(X_f) \cap E = \emptyset$ . In addition,  $\pi(X) \cap Y_f = \pi(X_f)$ , because  $X_f = \pi^{-1}(Y_f)$ . Hence

$$\emptyset = E \cap \pi(X_f) = E \cap \pi(X) \cap Y_f = \overline{E \cap \pi(X)} \cap Y_f,$$

and the claim follows.  $\square$

*Remark 3.* Let  $X$  be an irreducible affine variety with a  $\mathbb{C}^+$ -action. Then we have  $\text{Quot}(\mathcal{O}(X)^{\mathbb{C}^+}) = \text{Quot}(\mathcal{O}(X))^{\mathbb{C}^+}$ . In fact, let  $r \in \text{Quot}(\mathcal{O}(X))^{\mathbb{C}^+}$  be an invariant rational function. Then  $\mathfrak{a} := \{q \in \mathcal{O}(X) \mid qr \in \mathcal{O}(X)\}$  is a  $\mathbb{C}^+$ -stable ideal in  $\mathcal{O}(X)$  and so  $\mathfrak{a}^{\mathbb{C}^+} \neq 0$  which implies that  $r \in \text{Quot}(\mathcal{O}(X)^{\mathbb{C}^+})$ . More generally, if  $f \in \mathcal{O}(X)^{\mathbb{C}^+}$  is an invariant function then  $(\mathcal{O}(X)_f)^{\mathbb{C}^+} = (\mathcal{O}(X)^{\mathbb{C}^+})_f$ .

Assume now that  $E$  is not finite and let  $C_i$  be an irreducible component of  $C = \overline{E \cap \pi(X)}$  of dimension 1. Then  $S_i := \pi^{-1}(C_i)$  is irreducible and reduced of dimension 2,  $\pi|_{S_i}: S_i \rightarrow C_i$  is smooth (Proposition 1) and there are infinitely many fibers of  $\pi|_{S_i}$  which have more than one component. Thus we get a factorization

$$\pi|_{S_i} = \varphi_i \circ \pi_{S_i}: S_i \rightarrow S_i // \mathbb{C}^+ \rightarrow C_i$$

where  $\varphi_i: S_i // \mathbb{C}^+ \rightarrow C_i$  has degree  $> 1$ . It also follows that the fibers of the quotient morphism  $\pi_{S_i}$  are reduced, hence smooth. Now we have the following general result.

**Lemma 2.** *Let  $S_0$  be an irreducible affine surface with a non-trivial  $\mathbb{C}^+$ -action.*

- (a) *The quotient  $S_0 // \mathbb{C}^+$  is an irreducible curve and  $\pi: S_0 \rightarrow S_0 // \mathbb{C}^+$  is surjective.*
- (b) *There is a dense open set  $U \subset S_0 // \mathbb{C}^+$  and an isomorphism  $\mathbb{C}^+ \times U \xrightarrow{\sim} \pi^{-1}(U)$  over  $U$ .*
- (c) *Assume that the action is fixed point free and that the fibers of  $\pi$  are reduced. Then the following are equivalent:*
  - (i)  *$S_0$  is smooth.*
  - (ii)  *$S_0$  is normal.*
  - (iii)  *$S_0 // \mathbb{C}^+$  is normal, hence smooth.*

*If (i), (ii) or (iii) holds then  $\pi$  is smooth.*

*Proof.* (a) Assume that  $\pi$  is not surjective. Then there is a rational function  $r$  on  $S_0 // \mathbb{C}^+$  which is defined on  $\pi(S_0)$  but not on  $S_0 // \mathbb{C}^+$ . Then  $\pi^*(r) \in \mathcal{O}(S_0)^{\mathbb{C}^+}$  and so  $r \in \mathcal{O}(S_0 // \mathbb{C}^+)$ , contradicting the assumption.

(b) This is a general fact: For every non-trivial  $\mathbb{C}^+$ -action on an irreducible affine variety  $Z$  there is an invariant function  $f \neq 0$  and a closed subset  $T \subset Z_f$  such that the action induces an isomorphism  $\mathbb{C}^+ \times T \xrightarrow{\sim} Z_f$ .

(c) We clearly have the implications (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii). By assumption, the fibers of  $\pi$  are reduced and are finite unions of orbits, hence smooth. As a consequence, we have  $T_x(\mathbb{C}^+x) = \text{Ker}(d\pi)_x$  for every  $x \in S_0$ , and so  $S_0$  is smooth in  $x$  if and only if  $S_0//\mathbb{C}^+$  is smooth in  $\pi(x)$ , and in this case  $\pi$  is smooth in  $x$ .  $\square$

Let us go back to the situation above where  $C = \overline{E \cap \pi(X)} = \bigcup_i C_i$  and  $S_i = \pi^{-1}(C_i)$ , and consider the factorization as above:

$$S_i \xrightarrow{\pi_{S_i}} D_i := S_i//\mathbb{C}^+ \xrightarrow{\varphi_i} C_i$$

**Lemma 3.** *In the following commutative diagram the vertical maps  $\eta_{S_i}$  and  $\eta_{C_i}$  denote the normalizations. Then the outer square is Cartesian,  $\tilde{S}_i$  is smooth, the morphism  $\tilde{\varphi}_i: \tilde{S}_i//\mathbb{C}^+ \rightarrow \tilde{C}_i$  is smooth and  $\eta$  is birational.*

$$\begin{array}{ccccc} \tilde{S}_i & \xrightarrow{\pi_{\tilde{S}_i}} & \tilde{S}_i//\mathbb{C}^+ & \xrightarrow{\tilde{\varphi}_i} & \tilde{C}_i \\ \downarrow \eta_{S_i} & & \downarrow \eta & & \downarrow \eta_{C_i} \\ S_i & \xrightarrow{\pi_{S_i}} & S_i//\mathbb{C}^+ & \xrightarrow{\varphi_i} & C_i \end{array}$$

*Proof.* Since  $\pi|_{S_i}: S_i \rightarrow C_i$  is smooth the fiber product  $\tilde{S}_i := S_i \times_{C_i} \tilde{C}_i$  is a smooth irreducible surface and so the induce (finite and birational) morphism  $\tilde{S}_i \rightarrow S_i$  is the normalization. Therefore, the quotient  $\tilde{S}_i//\mathbb{C}^+$  is normal, hence smooth, and the composition  $\tilde{S}_i \xrightarrow{\pi_{\tilde{S}_i}} \tilde{S}_i//\mathbb{C}^+ \rightarrow \tilde{C}_i$  is smooth. It follows that  $\tilde{S}_i//\mathbb{C}^+ \rightarrow \tilde{C}_i$  is smooth (because  $\pi_{\tilde{S}_i}$  is surjective by Lemma 2(a)). Since  $\pi_{S_i}$  is the trivial  $\mathbb{C}^+$ -bundle over a dense open set (Lemma 2(b)) we see that  $\eta$  is birational.  $\square$

*Remark 4.* We believe that  $\eta: \tilde{S}_i//\mathbb{C}^+ \rightarrow S_i//\mathbb{C}^+$  is finite, i.e. that  $\eta$  is the normalization map.

For a general variety  $Z$  with irreducible components  $Z = \bigcup_i Z_i$  we denote by  $\tilde{Z}$  the disjoint union of the normalizations  $\tilde{Z}_i$  of the components  $Z_i$  and will call the canonical morphism  $\eta_Z: \tilde{Z} \rightarrow Z$  the normalization of  $Z$ . Now Lemma 3 immediately implies the following result.

**Lemma 4.** *In the following diagram where  $\eta_S$  and  $\eta_C$  are the normalizations the outer square is cartesian,  $\tilde{S}$  is smooth and the morphisms  $\pi_{\tilde{S}}$  and  $\tilde{\varphi}$  are both smooth.*

$$\begin{array}{ccccc} \tilde{S} & \xrightarrow{\pi_{\tilde{S}}} & \tilde{D} := \tilde{S}//\mathbb{C}^+ & \xrightarrow{\tilde{\varphi}} & \tilde{C} \\ \downarrow \eta_S & & \downarrow \eta & & \downarrow \eta_C \\ S & \xrightarrow{\pi_S} & D := S//\mathbb{C}^+ & \xrightarrow{\varphi} & C \end{array}$$

## §4. TOPOLOGICAL CONSIDERATIONS (FOLLOWING SHULIM KALIMAN)

We follow here the ideas of SHULIM KALIMAN [Ka04] to give a proof of the conjecture of §1, based on some topological considerations. In fact, it suffices to assume that the ambient variety  $X$  is smooth and contractible and that the quotient  $Y$  is smooth. (The latter is even automatically the case as shown in the subsequent paper [KaS04].)

**Main Theorem.** *Let  $X$  be a 3-dimensional smooth affine variety with a free  $\mathbb{C}^+$ -action. Assume that  $X$  is factorial, that  $H_2(X) = H_3(X) = 0$  and that  $Y := X//\mathbb{C}^+$  is smooth. Then the action is a translation, i.e.  $X$  is  $\mathbb{C}^+$ -isomorphic to  $\mathbb{C}^+ \times Y$ .*

We use the notation introduced in the previous section. Clearly, there is a finite subset  $\mathcal{F} \subset C$  such that the following holds:

- (1)  $C' := C \setminus \mathcal{F}$  is smooth,
- (2)  $D' := D \setminus \varphi^{-1}(\mathcal{F})$  is smooth,
- (3)  $\varphi' := \varphi|_{D'}: D' \rightarrow C'$  is unramified and surjective,
- (4)  $S' := S \setminus \pi^{-1}(\mathcal{F})$  is smooth and  $\pi_{S'}: S' \rightarrow D'$  is a trivial  $\mathbb{C}^+$ -bundle.

Put  $Y' := Y \setminus \mathcal{F}$ ,  $X' := X \setminus \pi^{-1}(\mathcal{F})$ ,  $Y^0 := Y \setminus C$  and  $X^0 := X \setminus S$ . Then  $X^0 \subset X' \subset X$  and  $Y^0 \subset Y' \subset Y$ , and  $\pi^0: X^0 \rightarrow \pi(X^0) \subset Y^0$  is a principal  $\mathbb{C}^+$ -bundle by Proposition 2. We get the following exact homology sequence where the vertical maps are induced by  $\pi$ :

$$\begin{array}{ccccccccc} \cdots & \rightarrow & H_{j+1}(X^0) & \rightarrow & H_{j+1}(X') & \rightarrow & H_{j+1}(X', X^0) & \rightarrow & H_j(X') & \rightarrow & H_j(X^0) & \rightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \rightarrow & H_{j+1}(Y^0) & \rightarrow & H_{j+1}(Y') & \rightarrow & H_{j+1}(Y', Y^0) & \rightarrow & H_j(Y') & \rightarrow & H_j(Y^0) & \rightarrow & \cdots \end{array}$$

In the sequel we will frequently use the Thom isomorphism in the following form (see [Do72, VIII.11]).

**Thom Isomorphism.** *Let  $Z$  be a smooth variety and  $A \subset Z$  a closed smooth subvariety of codimension  $d$ . Put  $Z^0 := Z \setminus A$ . Then there is an isomorphism*

$$H_j(Z, Z^0) \xrightarrow{\sim} H_{j-2d}(A).$$

*In particular,  $H_j(Z, Z^0) = 0$  for  $j < 2d$  and so  $H_j(Z^0) \xrightarrow{\sim} H_j(Z)$  for  $j < 2d - 1$ .*

Recall that the homology of an affine variety  $Z$  vanishes above  $\dim Z$  (see [Mi63]). In particular,  $H_j(Y) = H_j(Y^0) = 0$  for  $j > 2$  and  $H_j(X) = H_j(X^0) = 0$  for  $j > 3$ . Since  $\pi^{-1}(\mathcal{F})$  consists of a finite number of  $\mathbb{C}^+$ -orbits we have  $H_i(\pi^{-1}(\mathcal{F})) = 0$  for  $i > 0$  and therefore get, from the Thom isomorphism and the exact sequence of the pair  $(X, X')$ ,

$$H_i(X') \xrightarrow{\sim} H_i(X) \text{ for } i = 0, 1, 2 \quad \text{and} \quad H_j(X') = H_j(X) = 0 \text{ for } j \geq 4,$$

together with an exact sequence

$$0 \rightarrow H_0(\pi^{-1}(\mathcal{F})) \rightarrow H_3(X') \rightarrow H_3(X) \rightarrow 0.$$

Similarly,  $H_i(Y') \xrightarrow{\sim} H_i(Y)$  for  $i = 0, 1, 2$ ,  $H_j(Y') = H_j(Y) = 0$  for  $j \geq 4$  and  $H_0(\mathcal{F}) \simeq H_3(Y')$ . Moreover,  $H_j(X^0) \xrightarrow{\sim} H_j(\pi(X^0))$  for all  $j$  since  $X^0 \rightarrow \pi(X^0)$  is a principal  $\mathbb{C}^+$ -bundle. Hence  $H_2(X^0) \xrightarrow{\sim} H_2(\pi(X^0)) \xrightarrow{\sim} H_2(Y^0)$ , because  $Y^0 \setminus \pi(X^0)$  is a finite set. Finally,  $H_3(X', X^0) \xrightarrow{\sim} H_1(S') \simeq H_1(D')$  and  $H_3(Y', Y^0) \simeq H_1(C')$ .

Now assume that  $H_2(X) = 0$ . Then we get the following commutative diagram with exact rows where  $\theta$  is induced by  $\varphi: D \rightarrow C$ :

$$\begin{array}{ccccccccc}
 H_3(X^0) & \longrightarrow & H_3(X') & \xrightarrow{\rho_X} & H_1(D') & \xrightarrow{\mu_X} & H_2(X^0) & \longrightarrow & 0 \\
 (*) & & \downarrow & & \delta' \downarrow & & \theta \downarrow & & \simeq \downarrow & & \downarrow \\
 0 & \longrightarrow & H_3(Y') & \xrightarrow{\rho_Y} & H_1(C') & \xrightarrow{\mu_Y} & H_2(Y^0) & \longrightarrow & H_2(Y')
 \end{array}$$

It is well-known and easy to see that the normalization  $\eta_D: \tilde{D} \rightarrow D$  induces an injection  $H_1(\tilde{D}) \hookrightarrow H_1(D)$  so that the two natural maps

$$\iota_D: H_1(D') \rightarrow H_1(D) \quad \text{and} \quad \iota_{\tilde{D}}: H_1(D') \rightarrow H_1(\tilde{D})$$

have the same kernel. Moreover,  $\iota_{\tilde{D}}$  is surjective. Clearly, the same holds for the curve  $C$  and so, using again the Thom Isomorphism, we get the following diagram with exact rows:

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & H_0(\tilde{D} \setminus D') & \longrightarrow & H_1(D') & \xrightarrow{\iota_{\tilde{D}}} & H_1(\tilde{D}) & \longrightarrow & 0 \\
 (**) & & \theta_0 \downarrow & & \theta \downarrow & & \tilde{\theta} \downarrow & & \\
 0 & \longrightarrow & H_0(\tilde{C} \setminus C') & \longrightarrow & H_1(C') & \xrightarrow{\iota_{\tilde{C}}} & H_1(\tilde{C}) & \longrightarrow & 0
 \end{array}$$

In other words, the kernel of the homomorphism  $\iota_{\tilde{C}}$  and hence of  $\iota_C$  is generated by simple loops around the points of  $\tilde{C} \setminus C'$ . The crucial point is to show that  $\tilde{\theta}$  is an isomorphism.

**Proposition 3.** *Assume  $H_2(X) = H_3(X) = 0$ . Then  $C \subset \pi(X)$ . In particular,  $\varphi: D \rightarrow C$  and  $\tilde{\varphi}: \tilde{D} \rightarrow \tilde{C}$  are both surjective.*

*Proof.* If  $C \not\subset \pi(X)$  then there is a point  $y_0 \in \mathcal{F}$  which is not in  $\pi(X)$  (by assumption (3) on the set  $\mathcal{F}$ , see above). The isomorphism  $H_0(\mathcal{F}) \xrightarrow{\sim} H_3(Y')$  maps the point  $y_0$  to the class  $\omega$  given by a 3-sphere in  $Y'$  around  $y_0$ . The image  $\gamma_C := \rho_Y(\omega) \in H_1(C')$  is the sum of simple loops  $\gamma_i \subset \Gamma_i$  around  $y_0$  in the different analytic branches  $\Gamma_i$  of  $C$  through  $y_0$ . These loops can be lifted to  $D'$  which implies that there is an element  $\gamma_D \in H_1(D')$  such that  $\theta(\gamma_D) = m\gamma_C$  with an integer  $m > 0$ , see the diagram (\*) above. It follows that  $\mu_X(\gamma_D) = 0$  and so  $\gamma_D = \rho_X(\omega_X)$  for some  $\omega_X \in H_3(X')$  such that  $\delta'(\omega_X) = m\omega$ . Now we have the commutative diagram

$$\begin{array}{ccc}
 H_3(X') & \xrightarrow{\tau_X} & H_3(X) = 0 \\
 \delta' \downarrow & & \delta \downarrow \\
 H_3(Y') & \xrightarrow{\tau_Y} & H_3(\pi(X))
 \end{array}$$

By construction,  $\tau_Y(\omega) \neq 0$  since  $y_0 \notin \pi(X)$ , and so  $\tau_Y(\delta'(\omega_X)) \neq 0$  which leads to a contradiction. Thus  $\pi: X \rightarrow Y$  is surjective, as well as  $\varphi: D \rightarrow C$ . Now Lemma 4 implies that  $\tilde{\varphi}: \tilde{D} \rightarrow \tilde{C}$  is surjective, too.  $\square$

*Remark 5.* It follows from the diagram (\*\*) above and Proposition 3 that the homomorphism  $\theta: H_1(D') \rightarrow H_1(C')$  maps the kernel of  $\iota_{\tilde{D}}$  surjectively onto the kernel of  $\iota_{\tilde{C}}$ , because  $\tilde{D} \setminus D' \rightarrow \tilde{C} \setminus C'$  is surjective.

*Remark 6.* In case  $X = \mathbb{C}^3$  it was shown by BONNET in [Bo02] that the quotient map  $\pi$  is surjective which implies the result above in this case.

**Proposition 4.** *Assume that  $H_2(X) = H_3(X) = 0$ . Then, with the notation above,*

$$\begin{aligned} \rho_X(H_3(X')) &\subset \text{Ker}(\iota_D: H_1(D') \rightarrow H_1(D)), \\ \rho_Y(H_3(Y')) &\subset \text{Ker}(\iota_C: H_1(C') \rightarrow H_1(C)) \end{aligned}$$

(see the diagram (\*) above).

*Proof.* Let us first look at the map  $\rho_Y$  and recall that  $Y' = Y \setminus \mathcal{F}$ . The isomorphism  $H_0(\mathcal{F}) \xrightarrow{\sim} H_3(Y')$  says that  $H_3(Y')$  is generated by small spheres around the points of  $\mathcal{F}$ . Thus we get a sequence of maps

$$H_0(\mathcal{F}) \xrightarrow{\sim} H_3(Y') \xrightarrow{\rho_Y} H_3(Y', Y^0) \xrightarrow{\sim} H_1(C') \xrightarrow{\iota_C} H_1(C)$$

and we have to show that the composition is zero. For every  $y \in \mathcal{F}$  we choose a small open neighborhood  $U_y \subset Y$  such that  $U_y \cap C$  is contractible. Put  $U := \bigcup_y U_y$ ,  $U' := U \cap Y'$  and  $U^0 := U \cap Y^0 = U \setminus C$ . Using the functoriality of the Thom isomorphism with respect to open embeddings (see [Do72, VIII.11]) we obtain the following commutative diagram (using  $U \setminus U' = Y \setminus Y' = \mathcal{F}$ )

$$\begin{array}{ccccccc} H_0(U \setminus U') & \longrightarrow & H_3(U') & \longrightarrow & H_1(C' \cap U) & \longrightarrow & H_1(C \cap U) = 0 \\ \parallel & & \downarrow & & \downarrow & & \downarrow \\ H_0(\mathcal{F}) & \xrightarrow{\sim} & H_3(Y') & \xrightarrow{\rho_Y} & H_1(C') & \xrightarrow{\iota_C} & H_1(C) \end{array}$$

and the claim follows.

For  $X$  and the map  $\rho_X$  we get a similar diagram (using  $H_3(X) = 0$ ):

$$\begin{array}{ccccccc} H_0(X \setminus X') & \xrightarrow{\sim} & H_3(X') & \xrightarrow{\rho_X} & H_3(X', X^0) & \xrightarrow{\sim} & H_1(S') \longrightarrow H_1(S) \\ & & & & & & \simeq \downarrow \qquad \downarrow \\ & & & & & & H_1(D') \xrightarrow{\iota_D} H_1(D) \end{array}$$

We have to show that the composition  $H_0(X \setminus X') \rightarrow H_1(D)$  is zero. This time we choose for every orbit  $O \subset X \setminus X'$  a small open neighborhood  $U_O \subset X$  with the property that the image  $V_O$  of  $S \cap U_O$  in  $D$  is contractible. Setting  $U := \bigcup_O U_O$ ,

$U' := U \cap X'$ ,  $U^0 := U \cap X^0$  and  $V = \bigcup V_O$  we obtain, as above, the commutative diagram

$$\begin{array}{ccccccccc} H_0(U \setminus U') & \longrightarrow & H_3(U') & \longrightarrow & H_1(S' \cap U) & \longrightarrow & H_1(S \cap U) & \longrightarrow & H_1(V) = 0 \\ & & \parallel & & \downarrow & & \downarrow & & \downarrow \\ H_0(X \setminus X') & \xrightarrow{\simeq} & H_3(X') & \xrightarrow{\rho_X} & H_1(S') & \longrightarrow & H_1(S) & \longrightarrow & H_1(D) \end{array}$$

Hence, the horizontal composition of the maps is zero and the claim follows.  $\square$

*Proof of the Main Theorem.* We look again at the diagram (\*\*) and claim that  $\tilde{\theta}$  is an isomorphism. It follows from Proposition 4 that  $\iota_{\tilde{D}}$  factors through  $\mu_X$  and  $\iota_{\tilde{C}}$  through  $\mu_Y$  (see the diagram (\*)), inducing the diagram

$$\begin{array}{ccccccc} H_1(D') & \xrightarrow{\mu_X} & H_2(X^0) & \xrightarrow{\bar{\iota}_D} & H_1(\tilde{D}) & \longrightarrow & 0 \\ \theta \downarrow & & \simeq \downarrow & & \tilde{\theta} \downarrow & & \\ H_1(C') & \xrightarrow{\mu_Y} & H_2(Y^0) & \xrightarrow{\bar{\iota}_C} & H_1(\tilde{C}) & \longrightarrow & 0 \end{array}$$

where the composition of the horizontal maps are  $\iota_{\tilde{D}}$  and  $\iota_{\tilde{C}}$ , respectively. Since  $\theta$  maps the kernel of  $\iota_{\tilde{D}}$  surjectively onto the kernel of  $\iota_{\tilde{C}}$  (Remark 5) we see that the isomorphism  $H_2(X^0) \xrightarrow{\simeq} H_2(Y^0)$  maps the kernel of  $\bar{\iota}_D$  onto the kernel of  $\bar{\iota}_C$ . Hence  $\tilde{\theta}$  is an isomorphism, too.

By definition,  $H_1(\tilde{C}) = \bigoplus_i H_1(\tilde{C}_i)$  and  $H_1(\tilde{D}) = \bigoplus_i H_1(\tilde{D}_i)$  and so the (unramified) morphisms  $\tilde{D}_i \rightarrow \tilde{C}_i$  induce an isomorphisms  $H_1(\tilde{D}_i) \xrightarrow{\simeq} H_1(\tilde{C}_i)$  for all  $i$ . The next well-known lemma shows that  $\tilde{\varphi}$  is an isomorphism which contradicts our assumption that the set  $E$  is not finite. Therefore, by Proposition 2, the morphism  $\pi: X \rightarrow \pi(X)$  is a principal  $\mathbb{C}^+$ -bundle. The claim follows from Corollary 2 once we show that  $\pi(X) = Y$ . But this follows from the exact homology sequence for the pair  $(Y, \pi(X))$  and the Thom Isomorphism:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_4(Y) & \longrightarrow & H_4(Y, \pi(X)) & \longrightarrow & H_3(\pi(X)) & \longrightarrow & \cdots \\ & & \parallel & & \simeq \downarrow & & \parallel & & \\ & & 0 & & H_0(Y \setminus \pi(X)) & & 0 & & \end{array}$$

$\square$

**Lemma 5.** *Let  $C_0, D_0$  be irreducible affine smooth curves and  $\varphi: D_0 \rightarrow C_0$  an unramified surjective morphism. Assume that  $\varphi$  induces an isomorphism  $H_1(D_0) \xrightarrow{\simeq} H_1(C_0)$ . Then  $\varphi$  is an isomorphism.*

*Proof.* Assume that the morphism  $\varphi$  is of degree  $d > 1$ . Let  $\mathcal{F} \subset C_0$  be the (finite) subset of those points  $y$  where  $\#\varphi^{-1}(y) < d$ . Then we obtain the following relation between the Euler characteristics of the curves  $C_0$  and  $D_0$ :

$$\chi(D_0) - \#\varphi^{-1}(\mathcal{F}) = d(\chi(C_0) - \#\mathcal{F}).$$

By assumption,  $\chi(D_0) = \chi(C_0)$  and  $\#\varphi^{-1}(\mathcal{F}) \leq d \#\mathcal{F}$  and so  $(d-1)\chi(C_0) \geq 0$ . This implies that  $C_0$  and  $D_0$  are both isomorphic to  $\mathbb{C}$  or to  $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ . The first case cannot occur since there are no unramified morphisms  $\mathbb{C} \rightarrow \mathbb{C}$  of degree  $> 1$ . In the second case there is, up to isomorphism only one morphism  $\mathbb{C}^* \rightarrow \mathbb{C}^*$  of degree  $d$ , namely  $z \mapsto z^d$ , and this one induces the map  $d \cdot \text{Id}$  on  $H_1(\mathbb{C}^*) \simeq \mathbb{Z}$  which is not an isomorphism.  $\square$

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