

# COMPRESSION OF FINITE GROUP ACTIONS AND COVARIANT DIMENSION, II

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ABSTRACT. Let  $G$  be a finite group and  $\varphi: V \rightarrow W$  an equivariant polynomial map between finite dimensional  $G$ -modules. We say that  $\varphi$  is faithful if  $G$  acts faithfully on  $\overline{\varphi(V)}$ . The covariant dimension of  $G$  is the minimum of the dimension of  $\overline{\varphi(V)}$  taken over all faithful  $\varphi$ . In [KS07] we investigated covariant dimension and were able to determine it in many cases. Our techniques largely depended upon finding homogeneous faithful covariants. After publication of [KS07], the junior author of this article pointed out several gaps in our proofs. Fortunately, this inspired us to find better techniques, involving multihomogeneous covariants, which have enabled us to extend and complete the results, simplify the proofs and fill the gaps of [KS07].

## 1. INTRODUCTION

For simplicity we take our ground field to be the field  $\mathbb{C}$  of complex numbers. Much work has been done in the context of more general fields (see, for example, [Flo08, JLY02, Led07, KM08]). In [Löt08] our results are extended to this context. (See the remarks at the end of this section.)

Let  $G$  be a finite group. All  $G$ -modules that we consider will be finite dimensional. A *covariant* of  $G$  is an equivariant morphism (= polynomial map)  $\varphi: V \rightarrow W$  where  $V$  and  $W$  are  $G$ -modules. The *dimension* of  $\varphi$  is defined to be the dimension of the image of  $\varphi$ :

$$\dim \varphi := \dim \overline{\varphi(V)}.$$

The covariant  $\varphi$  is *faithful* if the group  $G$  acts faithfully on the image  $\varphi(V)$ . Equivalently, there is a point  $w \in \varphi(V)$  with trivial isotropy group  $G_w$ . The *covariant dimension*  $\text{covdim } G$  of  $G$  is defined to be the minimum of  $\dim \varphi$  where  $\varphi: V \rightarrow W$  runs over all faithful covariants of  $G$ . If  $\dim \varphi = \text{covdim } G$  we say that  $\varphi$  is a *minimal covariant*. In [KS07, Proposition 2.1] we show that there is a minimal covariant  $\varphi: V \rightarrow W$  if  $V$  and  $W$  are faithful. In particular, if  $V$  is a faithful  $G$ -module, then there is a minimal faithful covariant  $\varphi: V \rightarrow V$ .

Suppose that  $\varphi: V \rightarrow W$  is a *rational map* which is  $G$ -equivariant. We call  $\varphi$  a *rational covariant*. Then one can define the notion of  $\varphi$  being faithful and the dimension of  $\varphi$  as in the case of ordinary covariants. The *essential dimension*  $\text{edim } G$  of  $G$  is the minimum dimension of all its faithful rational covariants. It is easy to see that

$$\text{edim } G \leq \text{covdim } G \leq \text{edim } G + 1$$

(see [Rei04] or the proof of Theorem 2.6 below).

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Our results in [KS07] were largely based upon finding homogeneous minimal covariants. Unfortunately, this is not always possible [KS07, Remark 4.1]. In this paper, however, we are able to show that there are always *multihomogeneous* minimal covariants. This allows us to improve upon the results of [KS07]. In particular, we are able to obtain the exact relation between covariant and essential dimension (Theorem 3.1):

$$\text{covdim } G = \begin{cases} \text{edim } G + 1, & \text{if the center of } G \text{ is trivial} \\ \text{edim } G, & \text{otherwise.} \end{cases}$$

In certain cases we are able to describe the image of a covariant (Proposition 4.1) and deduce that for a *faithful* group  $G$  (i.e.,  $G$  admits an irreducible faithful representation) we have  $\text{covdim}(G \times \mathbb{Z}/p\mathbb{Z}) = \text{covdim } G + 1$  if and only if the prime  $p$  divides the order  $|Z(G)|$  of the center of  $G$ . This completes the analysis of [KS07, §5–6]. In the process we repair the proofs of Corollaries 6.1 and 6.2 of [KS07]. They are supposed to be corollaries of Proposition 6.1, but the hypotheses of the proposition are not fulfilled. In section 5 we give some examples of covariant dimensions of groups, in part generalizing [KS07, Proposition 6.2]. In sections 6 and 7 we repair two proofs, one concerning a characterization of faithful groups and their subgroups, and one about the classification of non-faithful groups of covariant dimension 2. In section 8 we list some minor errata from [KS07].

In the case of a general field  $k$ , our considerations must be limited to groups admitting a completely reducible faithful  $k$ -representation. We also need  $k$  to contain a primitive root of unity of order equal to the exponent of  $Z(G)$  for every group  $G$  appearing in our arguments. In [Löt08] this last restriction can sometimes be avoided. If  $k$  is finite, some arguments do not go through since there may exist no  $k$ -rational points with trivial stabilizer in the image of a faithful covariant.

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## 2. MULTIHOMOGENEOUS COVARIANTS

Let  $V = \bigoplus_{i=1}^n V_i$  and  $W = \bigoplus_{j=1}^m W_j$  be direct sums of vector spaces and let  $\varphi = (\varphi_1, \dots, \varphi_m): V \rightarrow W$  be a morphism where none of the  $\varphi_j$  are zero. We say that  $\varphi$  is *multihomogeneous of degree*  $A = (\alpha_{ji}) \in M_{m \times n}(\mathbb{Z})$  if, for an indeterminate  $s$ , we have

$$\varphi_j(v_1, \dots, sv_i, \dots, v_n) = s^{\alpha_{ji}} \varphi_j(v_1, \dots, v_n) \text{ for all } j = 1, \dots, m, i = 1, \dots, n.$$

Whenever we consider the degree matrix  $A$  of some  $\varphi$ , we are always tacitly assuming that  $\varphi_j \neq 0$  for all  $j$ .

We now give a way to pass from a general  $\varphi$  to the multihomogeneous case. For indeterminates  $s_1, \dots, s_n$ , we have  $\varphi_j(s_1 v_1, \dots, s_n v_n) = \sum_{\alpha} \varphi_j^{(\alpha)} s^{\alpha}$  for each  $j$ , where  $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$  and  $s^{\alpha} = s_1^{\alpha_1} \dots s_n^{\alpha_n}$ . If  $\beta \in \mathbb{R}^n$ , let  $\alpha \cdot \beta$  denote the usual inner product. Now suppose for illustration that  $m = 1$  and  $\dim W_1 = \dim V_i = 1$ ,  $i = 1, \dots, n$ . Then  $\varphi = \varphi_1$  is just a polynomial in  $n$  variables. If the entries of  $\beta$  are linearly independent over  $\mathbb{Q}$ , then we can assign to any polynomial  $\varphi$  its initial term (corresponding to the monomial  $\varphi^{(\alpha)}$  with the highest value of  $\alpha \cdot \beta$ ). In the yoga of Gröbner basis theory ([CLO07, Stu96]) we are assigning to each  $\varphi$  its initial term with respect to the weighted monomial ordering given by  $\beta$ . This initial term is well-defined for any  $\varphi$ . In our situation, we only need the initial

terms to be well-defined for a finite collection of polynomials and we can choose  $\beta \in \mathbb{N}^n$ .

Returning to the general case, let  $\beta \in \mathbb{N}^n$  and set  $h_j = \max\{\alpha \cdot \beta \mid \varphi_j^{(\alpha)} \neq 0\}$ ,  $j = 1, \dots, m$ . For  $r \in \mathbb{N}$  set  $\varphi_j^{(r)} = \sum_{\alpha \cdot \beta = r} \varphi_j^{(\alpha)}$ . Now we fix a  $\beta$  such that, for each  $r \in \mathbb{N}$  and each  $j$ ,  $\{\alpha \mid \alpha \cdot \beta = r \text{ and } \varphi_j^{(\alpha)} \neq 0\}$  has cardinality at most 1. Thus  $\varphi_j^{(r)}$  is zero or consists of one nonzero term  $\varphi_j^{(\alpha)}$ . Moreover,  $\varphi_j^{(h_j)} \neq 0$  for all  $j$  and  $\varphi_{\max} := (\varphi_1^{(h_1)}, \dots, \varphi_m^{(h_m)})$  is multihomogeneous. Note that the  $h_j$  and so  $\varphi_{\max}$  depend upon our choice of  $\beta$ .

- Remarks 2.1.* (1) If the  $V_i$  and  $W_j$  are  $G$ -modules and  $\varphi$  is equivariant, so are all the  $\varphi_j^{(\alpha)}$  and  $\varphi_{\max}$ . Note that no entry in  $\varphi_{\max}$  is zero since the same is true of  $\varphi$ .
- (2) If  $\varphi: V \rightarrow W$  is multihomogeneous of degree  $A = (\alpha_{ji})$  and  $\psi: W \rightarrow U = \bigoplus_{k=1}^{\ell} U_k$  is multihomogeneous of degree  $B = (\beta_{kj})$  and all components of  $\psi \circ \varphi$  are non-zero, then the composition  $\psi \circ \varphi: V \rightarrow U$  is multihomogeneous of degree  $BA$ .

Concerning  $\varphi_{\max}$  there is the following main result.

**Lemma 2.2.** *Let  $\varphi: V \rightarrow W$  be a morphism where no  $\varphi_j$  is zero. Then  $\dim \varphi_{\max} \leq \dim \varphi$ .*

*Proof.* Let  $\beta$  and  $h_1, \dots, h_m$  be as above. We have an action  $\lambda$  of  $\mathbb{C}^*$  on  $W$  where  $\lambda(t)(w) = (t^{h_1}w_1, \dots, t^{h_m}w_m)$  for  $w \in W$  and  $t \in \mathbb{C}^*$ . We also have an action  $\mu$  of  $\mathbb{C}^*$  on  $V$  by  $\mu(t)(v_1, \dots, v_n) = (t^{\beta_1}v_1, \dots, t^{\beta_n}v_n)$  where  $t \in \mathbb{C}^*$  and  $v \in V$ . Let  ${}^t\varphi(v)$  denote  $\lambda(t)(\varphi(\mu(t^{-1})(v)))$  for  $t \in \mathbb{C}^*$  and  $v \in V$ . Then  ${}^t\varphi(v) = \varphi_{\max}(v) + t\psi(t, v)$  for some morphism  $\psi: \mathbb{C} \times V \rightarrow W$ . Consider the morphism

$$\Phi: \mathbb{C} \times V \rightarrow \mathbb{C} \times W, \quad (t, v) \mapsto (t, {}^t\varphi(v))$$

where  ${}^0\varphi := \varphi_{\max}$ . Let  $Y$  denote  $\text{Im } \Phi$ . Let  $p: \overline{Y} \rightarrow \mathbb{C}$  be the morphism induced by the projection  $\mathbb{C} \times W \rightarrow \mathbb{C}$  where  $\overline{Y}$  denotes the closure of  $Y$ . Clearly, we have  $Y \cap (\{t\} \times W) = \{t\} \times \text{Im } {}^t\varphi$  for  $t \in \mathbb{C}$ . Then

$$\overline{Y} \cap (\mathbb{C}^* \times W) = \bigcup_{t \neq 0} \{t\} \times \lambda(t)X$$

where  $X := \overline{\text{Im } \varphi}$ , because the right hand side is closed in  $\mathbb{C}^* \times W$ . As a consequence, we get

$$\overline{Y} = \overline{\Phi(\mathbb{C}^* \times W)},$$

hence  $p^{-1}(t) = \{t\} \times \overline{\text{Im } {}^t\varphi}$  for  $t \neq 0$  and  $p^{-1}(0) \supset \{0\} \times \overline{\text{Im } \varphi_{\max}}$ . Since  $\overline{Y}$  is irreducible, it follows that  $\dim \varphi_{\max} \leq \dim \varphi$ .  $\square$

**Corollary 2.3.** *Let  $p: \overline{Y} \rightarrow \mathbb{C}$  be as in the proof above. If  $\dim \varphi = \dim \varphi_{\max}$  and  $\text{Im } \varphi$  is  $\mathbb{C}^*$ -stable, then  $p^{-1}(0)$  is  $\mathbb{C}^*$ -stable.*

*Proof.* The hypotheses imply that  $\{0\} \times \overline{\text{Im } \varphi_{\max}}$  is an irreducible component of  $p^{-1}(0)$ . Since  $\text{Im } \varphi$  is  $\mathbb{C}^*$ -stable, then so is  $\text{Im } {}^t\varphi$  for all  $t \neq 0$  which implies that  $\overline{Y}$  is stable under the  $\mathbb{C}^*$ -action  $\lambda \cdot (t, w) := (t, \lambda w)$  on  $\mathbb{C} \times W$ . It follows that  $p^{-1}(0)$  is  $\mathbb{C}^*$ -stable.  $\square$

**Theorem 2.4.** *Let  $G$  be a finite group and let  $V = \bigoplus_{i=1}^n V_i$  and  $W = \bigoplus_{j=1}^m W_j$  be faithful representations where the  $V_i$  and  $W_j$  are irreducible submodules. Then there is a minimal regular multihomogeneous covariant  $\varphi: V \rightarrow W$  all of whose components are nonzero.*

*Proof.* Let  $\varphi: V \rightarrow W$  be a minimal covariant. We can always arrange that for given  $v \in V$  and  $w \in W$ , both with trivial stabilizer in  $G$ , we have  $\varphi(v) = w$  (see [KS07, Proposition 2.1]. This is also proved in [Pop94, Theorem 7.1.12], cf. [BR97, Lemma 3.2a]). Thus we can assume that all components of  $\varphi$  are nonzero. Then  $\varphi_{\max}: V \rightarrow W$  is a multihomogeneous covariant,  $\dim \varphi_{\max} \leq \dim \varphi$  and  $\varphi_{\max}$  is faithful since all its components are non-zero [KS07, Lemma 4.1].  $\square$

**Corollary 2.5.** *Let  $V_i$  be a faithful irreducible representation of the group  $G_i$ ,  $i = 1, \dots, n$ . Then  $V = \bigoplus_{i=1}^n V_i$  is a faithful representation of  $G := G_1 \times \dots \times G_n$ , and there is a minimal multihomogeneous covariant  $\varphi: V \rightarrow V$ .*  $\square$

We want to prove similar results for a *rational* covariant  $\psi: V \rightarrow W$ . It is obvious how to extend the definitions of *minimal* and *multihomogeneous of degree  $A$*  to rational covariants where in this case the matrix  $A$  might contain negative entries.

**Theorem 2.6.** *Let  $G$  be a finite group and let  $V = \bigoplus_{i=1}^n V_i$  and  $W = \bigoplus_{j=1}^m W_j$  be faithful representations where the  $V_i$  and  $W_j$  are irreducible submodules. Then there is a minimal rational multihomogeneous covariant  $\psi: V \rightarrow W$  all of whose components are non-zero and which is of the form  $\psi = h^{-1}\varphi$  where  $h$  is a multihomogeneous invariant and  $\varphi: V \rightarrow W$  a multihomogeneous minimal regular covariant.*

*Proof.* Let  $\psi: V \rightarrow W$  be a minimal rational covariant. We can assume that all components of  $\psi$  are nonzero. There is a nonzero invariant  $f \in \mathcal{O}(V)^G$  such that  $f\psi$  is regular. Define the regular covariant

$$\varphi := (f\psi, f): V \rightarrow W \oplus \mathbb{C}, \quad v \mapsto (f\psi(v), f(v))$$

which is faithful since  $\psi$  is. Moreover, either  $\dim \varphi = \overline{\dim \psi}$  or  $\dim \varphi = \dim \psi + 1$ , where the second case takes place if and only if  $\overline{\varphi(V)}$  is stable under scalar multiplication with  $\mathbb{C}^*$ . This follows from the fact that the composition of rational maps  $V \rightarrow W \oplus \mathbb{C} \rightarrow \mathbb{P}(W \oplus \mathbb{C}) \rightarrow W$  is  $\psi$ .

As above we obtain a multihomogeneous covariant  $\varphi_{\max}: V \rightarrow W \oplus \mathbb{C}$  which has the form  $\varphi_{\max} = (\varphi_1, \dots, \varphi_m, h)$ . Now define the multihomogeneous rational covariant

$$\psi_{\max} := \left( \frac{\varphi_1}{h}, \dots, \frac{\varphi_m}{h} \right): V \rightarrow W$$

which is again faithful. Moreover,  $\dim \psi_{\max} \leq \dim \varphi_{\max} \leq \dim \varphi$ . So if  $\dim \varphi = \dim \psi$  then  $\psi_{\max}$  is a minimal multihomogeneous rational covariant and we are done.

Now assume that  $\dim \varphi = \dim \psi + 1$  so that  $\overline{\varphi(V)}$  is  $\mathbb{C}^*$ -stable. If  $\varphi$  is not minimal then there is a minimal homogeneous regular covariant  $\tilde{\varphi}$  of dimension  $\leq \dim \psi$  and we are again done. Therefore we can assume that  $\varphi$  is minimal, hence  $\dim \varphi_{\max} = \dim \varphi$ . Since  ${}^t \overline{\varphi(V)}$  is  $\mathbb{C}^*$ -stable for all  $t \neq 0$  it follows from Corollary 2.3 that  $\overline{\varphi_{\max}(V)}$  is  $\mathbb{C}^*$ -stable, too, and so

$$\dim \psi_{\max} \leq \dim \varphi_{\max} - 1 = \dim \varphi - 1 = \dim \psi.$$

Hence,  $\psi_{\max}$  is a minimal multihomogeneous rational covariant.  $\square$

3. COVARIANT DIMENSION AND ESSENTIAL DIMENSION

In this section we extend [KS07, Corollary 4.2] to arbitrary groups and give the exact relation between covariant and essential dimension of finite groups.

**Theorem 3.1.** *Let  $G$  be a non-trivial finite group. Then  $\text{covdim } G = \text{edim } G$  if and only if  $G$  has a non-trivial center.*

The proof is given in Corollary 3.5 and Proposition 3.6 below. We need some preparation. In this section we have faithful representations  $V = \bigoplus_{i=1}^n V_i$  and  $W = \bigoplus_{j=1}^m W_j$  where the  $V_i$  and  $W_j$  are irreducible submodules. We have a natural action of the tori  $\mathbb{C}^{*n}$  on  $V$  and  $\mathbb{C}^{*m}$  on  $W$ . These actions are free on the open sets  $V' := \{v = (v_1, \dots, v_n) \mid v_i \neq 0 \text{ for all } i\} \subset V$  and  $W' \subset W$  defined similarly. If  $\varphi: V \rightarrow W$  is multihomogeneous of degree  $A = (\alpha_{ji})$  then  $\varphi$  is equivariant with respect to the homomorphism

$$T(A): \mathbb{C}^{*n} \rightarrow \mathbb{C}^{*m}, \quad s = (s_1, \dots, s_n) \mapsto (s^{\alpha_1}, s^{\alpha_2}, \dots, s^{\alpha_m})$$

where  $\alpha_j := (\alpha_{j1}, \alpha_{j2}, \dots, \alpha_{jn})$  and  $s^{\alpha_j} = s_1^{\alpha_{j1}} s_2^{\alpha_{j2}} \cdots s_n^{\alpha_{jn}}$ , as before. This implies that the (closure of the) image of  $\varphi$  is stable under the subtorus  $\text{Im } T(A) \subset \mathbb{C}^{*m}$ . The actions of  $G$  and  $\mathbb{C}^{*n}$  commute and since each  $V_i$  is irreducible, considered as subgroups of  $\text{GL}(V)$ , we have  $\mathbb{C}^{*n} \cap G = Z(G)$ .

*Remark 3.2.* Let  $\varphi: V \rightarrow W$  be a multihomogeneous covariant of degree  $A$ . If  $\mu \in A\mathbb{Q}^n \cap \mathbb{Z}^m$ , then  $\varphi(V) \subset W$  is stable under the  $\mathbb{C}^*$ -action  $\rho(t)(w_1, \dots, w_m) := (t^{\mu_1} w_1, \dots, t^{\mu_m} w_m)$ . It follows that for any invariant  $f \in \mathcal{O}(V)^G$  the morphism

$$(1) \quad \tilde{\varphi}: v = (v_1, \dots, v_n) \mapsto (f(v)^{\mu_1} \varphi_1(v), \dots, f(v)^{\mu_m} \varphi_m(v))$$

is a covariant with  $\tilde{\varphi}(V) \subset \varphi(V)$ , hence  $\dim \tilde{\varphi} \leq \dim \varphi$ . Moreover, if  $\varphi$  is faithful and  $f$  multihomogeneous, then  $\tilde{\varphi}$  is faithful and multihomogeneous of degree  $\tilde{A} := \mu \deg f + A$ , i.e.,  $\tilde{\alpha}_{ji} = \mu_j \deg_{V_i} f + \alpha_{ji}$ .

This has the following application which will be used later in the proof of Corollary 4.4: *Let  $p$  be a prime which does not divide the order of the center of  $G$ . Then there is a minimal multihomogeneous covariant  $\varphi: V \rightarrow V$  of degree  $A \not\equiv 0 \pmod{p}$ . (Start with a minimal multihomogeneous covariant  $\varphi: V \rightarrow V$  of degree  $A$  and assume that  $A \equiv 0 \pmod{p}$ . We can choose a  $\mu \in A\mathbb{Q}^n \cap \mathbb{Z}^m$  such that  $\mu_{j_0} \not\equiv 0 \pmod{p}$  for at least one  $j_0$ . Moreover, there is a multihomogeneous invariant  $f$  of total degree  $\not\equiv 0 \pmod{p}$  (see [KS07, Lemma 4.3]). But then  $\mu \deg f \not\equiv 0 \pmod{p}$ , and so the covariant  $\tilde{\varphi}$  given in (1) is minimal and has degree  $\mu \deg f + A \not\equiv 0 \pmod{p}$ .)*

For the next results we need some preparation. Let  $\varphi: V \rightarrow W$  be a multihomogeneous faithful covariant of degree  $A = (\alpha_{ji})$  where all components  $\varphi_j$  are non-zero. Define  $W' := \{(w_1, \dots, w_m) \in W \mid w_i \neq 0 \text{ for all } i\} = \prod_{j=1}^m (W_j \setminus \{0\})$ . The group  $\mathbb{C}^{*m}$  acts freely on  $W'$  and  $W' \rightarrow \prod_{j=1}^m \mathbb{P}(W_j)$  is the geometric quotient. Let  $X := \overline{\varphi(V)}$  and  $\mathbb{P}(X) \subset \prod_{j=1}^m \mathbb{P}(W_j)$  the image of  $X$ , and set  $X' := X \cap W'$ . Finally, denote by  $S \subset \mathbb{C}^{*m}$  the image of the homomorphism  $T(A): \mathbb{C}^{*n} \rightarrow \mathbb{C}^{*m}$ . Then we have the following.

**Lemma 3.3.** (1)  $\dim \mathbb{P}(X) \leq \dim X - \dim S \leq \dim X - \text{rank } Z(G)$ .  
 (2) *The kernel of the action of  $G$  on  $\mathbb{P}(X)$  is equal to  $Z(G)$ .*

*Proof.* We may regard  $G$  as a subgroup of  $\prod_{i=1}^m \mathrm{GL}(V_i)$  and of  $\prod_{j=1}^m \mathrm{GL}(W_j)$ , and so  $Z(G) = G \cap \mathbb{C}^{*n}$  and  $Z(G) = G \cap \mathbb{C}^{*m}$ .

(1) The first inequality is clear because  $X$  is stable under  $S$ . For the second we remark that  $Z(G) \subset S$  since  $\varphi$  is  $G$ -equivariant and so  $T(A)z = z$  for all  $z \in Z(G)$ .

(2) Let  $g \in G$  act trivially on  $\mathbb{P}(X)$ . Then every  $x \in X_j := \mathrm{pr}_{W_j}(X)$  is an eigenvector of  $g|_{W_j}$ . But  $X_j$  is irreducible and therefore contained in a fixed eigenspace of  $g$  on  $W_j$ . Since  $W_j$  is a simple  $G$ -module this implies that  $g|_{W_j}$  is a scalar.  $\square$

**Proposition 3.4.** *Let  $\varphi: V \rightarrow W$  be a multihomogeneous faithful covariant of degree  $A = (\alpha_{ji})$  where all components  $\varphi_j$  are non-zero. Assume that  $G$  has a trivial center. Then*

$$\mathrm{edim} G \leq \dim \varphi - \mathrm{rank} A \text{ and } \mathrm{covdim} G \leq \dim \varphi - \mathrm{rank} A + 1.$$

*In particular, if  $\varphi$  is a minimal regular covariant, then  $\mathrm{rank} A = 1$ , and if  $\varphi$  is a minimal rational covariant, then  $A = 0$ .*

*Proof.* Let  $X := \overline{\varphi(V)}$ , let  $\mathbb{P}(X) \subset \prod_{j=1}^m \mathbb{P}(W_j)$  denote the image of  $X$  and set  $X' := X \cap W'$ . Finally, let  $S$  denote the image of  $T(A): \mathbb{C}^{*n} \rightarrow \mathbb{C}^{*m}$ . The torus  $S$  has dimension  $\mathrm{rank} A$  and acts generically freely on  $X := \overline{\varphi(V)}$  since all components of  $\varphi$  are non-zero. Composing  $\varphi$  with the projection  $p: W \rightarrow \mathbb{P}(W_1) \times \cdots \times \mathbb{P}(W_m)$  we obtain a rational  $G$ -equivariant map  $\varphi': V \rightarrow \mathbb{P}(W_1) \times \cdots \times \mathbb{P}(W_m)$  such that  $\overline{\varphi'(V)} = \mathbb{P}(X)$ . Since  $Z(G)$  is trivial,  $G$  acts faithfully on  $\mathbb{P}(X)$ , and  $\dim \mathbb{P}(X) \leq \dim X - \dim S$ , by Lemma 3.3. Thus  $p \circ \varphi'$  is a rational faithful covariant of dimension  $\leq \dim X - \mathrm{rank} A$ , proving the first claim. The second follows since  $\mathrm{covdim} G \leq \mathrm{edim} G + 1$ .  $\square$

**Corollary 3.5.** *If  $G$  is a (non-trivial) group with trivial center, then*

$$\mathrm{covdim} G = \mathrm{edim} G + 1.$$

*Proof.* Let  $\varphi: V \rightarrow V$  be a minimal multihomogeneous regular covariant of degree  $A$ . By Proposition 3.4,  $\mathrm{rank} A = 1$  and  $\varphi$  is not minimal as a multihomogeneous rational covariant. Hence  $\mathrm{edim} G < \dim \varphi = \mathrm{covdim} G$  and the claim follows.  $\square$

**Proposition 3.6.** *If  $G$  has a non-trivial center, then  $\mathrm{covdim} G = \mathrm{edim} G$ .*

*Proof.* Let  $\psi: V \rightarrow V$  be a multihomogeneous minimal rational covariant of degree  $A = (\alpha_{ji})$  which is of the form  $h^{-1}\varphi$  where  $h \in \mathcal{O}(V)^G$  is a multihomogeneous invariant and  $\varphi: V \rightarrow V$  a multihomogeneous regular minimal covariant (Theorem 2.6).

(a) If there is a  $\beta \in \mathbb{Z}^n$  such that all entries of  $\gamma := A\beta$  are  $> 0$ , then the covariant  $\varphi := (h^{\gamma_1}\psi_1, \dots, h^{\gamma_n}\psi_n): V \rightarrow V$  is regular and faithful. Moreover,  $\overline{\varphi(V)} \subset \overline{\psi(V)}$  because the latter is stable under  $T(A)(\mathbb{C}^{*n})$ . Hence  $\mathrm{covdim} G \leq \dim \varphi \leq \dim \psi = \mathrm{edim} G$  and we are done.

(b) In general,  $A \neq 0$ , since otherwise the center of  $G$  would act trivially on the image  $\psi(V)$ . If  $\alpha_{j_0 i_0} \neq 0$ , choose a homogeneous invariant  $f \in \mathcal{O}(V_{j_0}) \subset \mathcal{O}(V)$  which does not vanish on  $\psi(V)$ . For any  $r \in \mathbb{Z}$  the composition  $\psi' := (f^r \cdot \mathrm{Id}) \circ \psi$  is still faithful and rational, and  $\dim \psi' \leq \dim \psi$ . Moreover, we get  $\psi'_j(v) = f^r(\psi_{j_0}(v)) \cdot \psi_j(v)$ . Therefore the degree of  $\psi'_j$  in  $V_{i_0}$  is  $r \cdot \deg f \cdot \alpha_{j_0 i_0} + \alpha_{j i_0}$  for  $j = 1, \dots, n$ . Hence, for a suitable  $r$ , all these degrees are  $> 0$ , and we are in case (a) with  $\beta := e_{i_0}$ .  $\square$

In some of our applications we will need the following result.

**Corollary 3.7.** *Assume that the center  $Z(G)$  is cyclic (and non-trivial) and that  $Z(G) \cap (G, G) = \{e\}$ . If  $G/Z(G)$  is faithful, then  $G$  is faithful, too, and*

$$\text{edim } G = \text{covdim } G = \text{covdim } G/Z(G) = \text{edim } G/Z(G) + 1.$$

*Proof.* It easily follows from the assumption  $Z(G) \cap (G, G) = \{e\}$  that the center of  $G/Z(G)$  is trivial and that every character of  $Z(G)$  can be lifted to a character of  $G$ . Now let  $V$  be an irreducible faithful representation of  $G/Z(G)$  and let  $\varphi: V \rightarrow V$  be a homogeneous minimal covariant. Since  $G/Z(G)$  has a trivial center we may assume that the degree of  $\varphi$  is  $\equiv 1 \pmod{|Z(G)|}$  (see Remark 3.2). If  $\chi: G \rightarrow \mathbb{C}^*$  is a character which is faithful on  $Z(G)$  then  $V \otimes \chi$  is an irreducible faithful representation of  $G$  and  $\varphi: V \otimes \chi \rightarrow V \otimes \chi$  is  $G$ -equivariant and faithful. Hence  $\text{covdim } G = \text{covdim } G/Z(G)$ . The other two equalities follow with Proposition 3.6 and Corollary 3.5.  $\square$

#### 4. THE IMAGE OF A COVARIANT

In certain cases one can get a handle on the ideal of  $\text{Im } \varphi$ .

**Proposition 4.1.** *Let  $V := \bigoplus_{i=1}^n V_i$  and let  $\varphi = (\varphi_1, \dots, \varphi_n): V \rightarrow V$  be a multihomogeneous morphism of degree  $A = (\alpha_{ji})$ . Assume that  $\det A \neq 0$ . Then the ideal  $\mathcal{I}(\varphi(V))$  of the image of  $\varphi$  is generated by multihomogeneous polynomials.*

*Proof.* For  $v = (v_1, \dots, v_n) \in V$  we have

$$\varphi(s_1 v_1, \dots, s_n v_n) = (s^{\alpha_1} \varphi_1, \dots, s^{\alpha_n} \varphi_n)(v)$$

where  $s^{\alpha_j} = s_1^{\alpha_{j1}} \dots s_n^{\alpha_{jn}}$ . Choose coordinates in each  $V_i$  and let  $M$  be a monomial in these coordinates. Let  $\beta = \beta(M)$  denote the multidegree of  $M$ , so we have  $M(s_1 v_1, \dots, s_n v_n) = s^\beta M(v_1, \dots, v_n)$ . Then  $M(\varphi(s_1 v_1, \dots, s_n v_n))$  is  $M(\varphi(v))$  multiplied by

$$(s^{\alpha_1}, \dots, s^{\alpha_n})^\beta = s_1^{\beta_1 \alpha_{11} + \dots + \beta_n \alpha_{n1}} \dots s_n^{\beta_1 \alpha_{1n} + \dots + \beta_n \alpha_{nn}} = s^{\beta A}$$

where  $\beta A$  is the matrix product of  $\beta$  and  $A$ . If  $F \in \mathcal{I}(\varphi(V))$ , we may write  $F = \sum_M c_M M$  where the  $c_M$  are constants and  $M$  varies over all monomials in the coordinates of the  $V_i$ . We have  $F(\varphi(s_1 v_1, \dots, s_n v_n)) = \sum_M c_M s^{\beta(M)A} M(\varphi(v))$ . Hence, for any  $\gamma \in \mathbb{N}^n$ , we obtain

$$\sum_{\beta(M)A=\gamma} c_M M \in \mathcal{I}(\varphi(V)).$$

Since  $\det A \neq 0$ , for any  $\gamma$  there is at most one  $\beta$  such that  $\beta A = \gamma$ . It follows that every sum of the form  $\sum_{\beta(M)=\beta} c_M M$  belongs to  $\mathcal{I}(\varphi(V))$ . Thus  $\mathcal{I}(\varphi(V))$  is generated by multihomogeneous polynomials.  $\square$

**Corollary 4.2.** *Suppose that  $\varphi$  is as above and that there is a  $k$ ,  $1 \leq k < n$ , such that  $\dim V_{k+1} = \dots = \dim V_n = 1$ . Then  $\dim \varphi = \dim(\varphi_1, \dots, \varphi_k) + (n - k)$ .*

*Proof.* Since the degree matrix  $A = (\alpha_{ji})$  exists, no  $\varphi_j$  is zero. Let  $m = \dim V_1 + \dots + \dim V_k$ . By Proposition 4.1 the ideal of  $\overline{\varphi(V)}$  is generated by functions of the form  $F(y_1, \dots, y_m) t_{k+1}^{r_{k+1}} \dots t_n^{r_n}$  where  $F$  is multihomogeneous. Such a function vanishes on  $\text{Im } \varphi$  if and only if  $F(y_1, \dots, y_m)$  vanishes on the image of  $(\varphi_1, \dots, \varphi_k)$ . Thus the ideal  $\mathcal{I}(\varphi(V))$  is generated by functions not involving  $t_{k+1}, \dots, t_n$ . As a consequence,  $\overline{\varphi(V)} = \overline{(\varphi_1, \dots, \varphi_k)(V)} \times V_{k+1} \times \dots \times V_n$ .  $\square$

In order to apply Proposition 4.1 and Corollary 4.2 we need a version of [KS07, Lemma 5.2].

**Corollary 4.3.** *Let  $G = G_1 \times \cdots \times G_n$  and  $V = V_1 \oplus \cdots \oplus V_n$  where each  $V_i$  is an irreducible representation of  $G_i$ ,  $i = 1, \dots, n$ . Let  $\varphi: V \rightarrow V$  be a multihomogeneous covariant of degree  $A$  and suppose that the prime  $p$  divides  $|Z(G_i)|$  for all  $i$ . Then  $\det A \neq 0$ , and the ideal  $\mathcal{I}(\varphi(V))$  is generated by multihomogeneous elements.*

*Proof.* Let  $\xi$  be a primitive  $p$ th root of unity. Then we have  $\varphi_j(v_1, \dots, \xi v_i, \dots, v_n) = \xi^{\alpha_{ji}} \varphi_j(v_1, \dots, v_n)$ . There is an element of  $G_j$  which acts as  $\xi$  on  $V_j$  and trivially on  $V_i$  if  $i \neq j$ . Hence  $\xi^{\alpha_{ji}} = 1$  for  $i \neq j$ . If  $i = j$ , one similarly shows that  $\xi^{\alpha_{jj}} = \xi$  by equivariance relative to  $G_j$ . This implies that

$$\alpha_{ji} \equiv \begin{cases} 1 \pmod{p} & \text{for } i = j, \\ 0 \pmod{p} & \text{otherwise,} \end{cases}$$

and so  $\det(\alpha_{ij}) \neq 0$ . Now apply Proposition 4.1.  $\square$

We say that  $G$  is *faithful* if it admits a faithful irreducible representation. We now get the following result which extends Corollaries 6.1 and 6.2 of [KS07].

**Corollary 4.4.** *Let  $G = G_1 \times \cdots \times G_n$  be a product of non-trivial faithful groups and let  $p$  be a prime.*

- (1) *If  $p$  is coprime to  $|Z(G)|$ , then  $\text{codim}(G \times \mathbb{Z}/p) = \text{codim } G$ .*
- (2) *If  $p$  divides all  $|Z(G_i)|$ , then  $\text{codim}(G \times (\mathbb{Z}/p)^m) = \text{codim } G + m$ .*

*In particular, if  $H$  is a non-trivial faithful group and  $m \geq 1$ , then*

$$\text{codim}(H \times (\mathbb{Z}/p)^m) = \begin{cases} \text{codim } H + m & \text{if } p \text{ divides } |Z(H)|; \\ \text{codim } H + (m - 1) & \text{otherwise.} \end{cases}$$

*Proof.* Let  $V_i$  be a faithful irreducible representation of  $G_i$ . Then  $V := V_1 \oplus \cdots \oplus V_n$  is a faithful representation of  $G$ . By Corollary 2.5 there is a minimal multihomogeneous faithful covariant  $\varphi = (\varphi_1, \dots, \varphi_k): V \rightarrow V$  of degree  $A$ . For any  $\delta = (\delta_1, \dots, \delta_n) \in \mathbb{Z}^n$  there is a linear action of  $\mathbb{Z}/p$  on  $V$  where the generator  $\bar{1} \in \mathbb{Z}/p$  acts by

$$v = (v_1, \dots, v_n) \mapsto (\zeta^{\delta_1} v_1, \dots, \zeta^{\delta_n} v_n), \quad \zeta := e^{\frac{2\pi i}{p}}.$$

This action commutes with the  $G$ -action and defines a  $G \times \mathbb{Z}/p$ -module structure on  $V$  which will be denoted by  $V_\delta$ . It follows that for  $\mu = A\delta$  the multihomogeneous map  $\varphi$  is a  $G \times \mathbb{Z}/p$ -equivariant morphism  $\varphi: V_\delta \rightarrow V_\mu$ . If  $p$  is coprime to  $|Z(G)|$  we can assume that  $A \not\equiv 0 \pmod{p}$  (Remark 3.2). Then there is a  $\delta$  such that  $\mu = A\delta \not\equiv 0 \pmod{p}$  and so  $\varphi$  is a faithful covariant for the group  $G \times \mathbb{Z}/p$ , proving (1).

Assume now that  $p$  divides all  $|Z(G_i)|$ . There is a minimal multihomogeneous covariant  $\psi: V \oplus \mathbb{C}^m \rightarrow V \oplus \mathbb{C}^m$  for  $G \times (\mathbb{Z}/p)^m$  where  $(\mathbb{Z}/p)^m$  acts in the obvious way on  $\mathbb{C}^m$ . Clearly, no entry of  $\psi$  is zero and by Corollaries 4.3 and 4.2, we get  $\dim \psi = \dim \varphi + m$  where  $\varphi: V \oplus \mathbb{C}^m \rightarrow V$  is  $\psi$  followed by projection to  $V$ . Since each component of  $\varphi$  is nonzero,  $\varphi$  is faithful for  $G$  [KS07, Lemma 4.1]. Thus  $\dim \varphi \geq \text{codim } G$ . But clearly,  $\text{codim}(G \times (\mathbb{Z}/p)^m) \leq \text{codim } G + m$ , hence we have equality, proving (2).  $\square$

As an immediate consequence we get the following result.

**Corollary 4.5.** *Let  $G$  be abelian of rank  $r$ . Then  $\text{covdim } G = r$ .*

*Remark 4.6.* The corollary is Theorem 3.1 of [KS07]. The proof in [KS07] uses a lemma whose proof is incorrect. The problem is that the quotient ring  $R/pR$  constructed there may have zero divisors. However, one can give a correct proof of the lemma by paying attention to the powers of the variables that occur in the determinant  $\det(\partial f_i/\partial x_j)$ . We omit this proof since the lemma is no longer needed.

The following strengthens [KS07, Proposition 6.1], which in turn then simplifies other proofs in the paper, e.g., the proof of Proposition 6.2.

**Corollary 4.7.** *Let  $V = W \oplus \mathbb{C}_\chi$  be a faithful representation of  $G$  where  $W$  is irreducible and  $\chi$  is a character of  $G$ . Let  $H$  denote the kernel of  $G \rightarrow \text{GL}(W)$ . Assume that there is a prime  $p$  which divides the order of  $H$  and such that the following two equivalent conditions hold:*

- (i) *There is a subgroup of  $\ker \chi$  acting as scalar multiplication by  $\mathbb{Z}/p$  on  $W$ ;*
- (ii) *There is a subgroup of  $G$  acting as scalar multiplication by  $\mathbb{Z}/p$  on  $V$ .*

*Then  $\text{covdim } G = \text{covdim } G/H + 1$ .*

*Proof.* It is easy to see that the two conditions are equivalent, because  $\chi|_H: H \rightarrow \mathbb{C}^*$  is injective. Since  $G$  embeds into  $G/H \times \chi(G)$ , we have  $\text{covdim } G \leq \text{covdim } G/H + 1$ .

To prove the reverse inequality let  $(\varphi, h): W \oplus \mathbb{C}_\chi \rightarrow W \oplus \mathbb{C}_\chi$  be a minimal faithful multihomogeneous covariant of degree  $\deg(\varphi, h) = (\alpha_{ji})$ . Since  $H$  is nontrivial,  $h$  cannot be zero. By assumption,  $H$  contains a subgroup of order  $p$  which is mapped injectively into  $\mathbb{C}^*$  by  $\chi$ . Thus the subgroup acts trivially on  $W$  and by scalar multiplication on  $\mathbb{C}_\chi$ . Therefore,

$$\alpha_{22} \equiv 1 \text{ and } \alpha_{12} \equiv 0 \pmod{p}.$$

Similarly, condition (i) implies that

$$\alpha_{11} \equiv 1 \text{ and } \alpha_{21} \equiv 0 \pmod{p}.$$

Thus  $\det(\alpha_{ij}) \neq 0$ , and so  $\dim(\varphi, h) = \dim \varphi + 1$  by Corollary 4.2. The equivariant morphism  $\varphi: W \oplus \mathbb{C}_\chi \rightarrow W$  factors through the quotient  $(W \oplus \mathbb{C}_\chi)/H$  which is isomorphic to the  $G/H$ -module  $W \oplus \mathbb{C}$ , and defines a faithful  $G/H$ -covariant  $\bar{\varphi}: W \oplus \mathbb{C} \rightarrow W$ . Hence,  $\dim \varphi \geq \text{covdim } G/H$ , and our result follows.  $\square$

Now consider the following commutative diagram with exact rows where  $\ell > m \geq 0$ ,  $\mu_N \subset \mathbb{C}^*$  denotes the  $N$ -th roots of unity and  $\pi$  is the canonical homomorphism  $\xi \mapsto \xi^{p^{\ell-m}}$ :

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \longrightarrow & G & \xrightarrow{\chi} & \mu_{p^\ell} & \longrightarrow & 1 \\ & & \parallel & & \downarrow & & \pi \downarrow & & \\ 1 & \longrightarrow & K & \longrightarrow & G' & \xrightarrow{\chi'} & \mu_{p^m} & \longrightarrow & 1 \end{array}$$

**Corollary 4.8.** *In the diagram above assume that  $G'$  is faithful and that the prime  $p$  divides  $|Z(G') \cap K|$ . Then  $\text{covdim } G = \text{covdim } G' + 1$ .*

*Proof.* Let  $\rho: G' \rightarrow \text{GL}(W)$  be a faithful irreducible representation. Then  $V := W \oplus \mathbb{C}_\chi$  is a faithful representation of  $G$ . Fix a  $p$ -th root of unity  $\zeta \in \mathbb{C}^*$  and let  $z' \in Z(G') \cap K$  be such that  $\rho(z') = \zeta \cdot \text{Id}$ . We have

$$G = \{(g', \xi) \in G' \times \mu_{p^\ell} \mid \chi'(g') = \pi(\xi)\}$$

and so  $z := (z', \zeta) \in Z(G)$  acts as scalar multiplication with  $\zeta$  on  $V$ . Now the claim follows from Corollary 4.7.  $\square$

## 5. SOME EXAMPLES

We consider the covariant dimension of some products and semidirect products of groups. We denote by  $C_n$  a cyclic group of order  $n$ .

**Example 5.1.** Consider the group  $G := C_3 \rtimes C_4$  where a generator of  $C_4$  acts on  $C_3$  by sending each element to its inverse. Then  $Z(G) \subset C_4$  is of order 2,  $(G, G) = C_3$  and  $G/Z(G) \simeq S_3$ . Hence  $\text{edim } G = \text{covdim } G = \text{covdim } S_3 = 2$ , by Corollary 3.7.

**Example 5.2.** Let  $H := S_3 \times S_3$ . Since  $\text{covdim } S_3 = 2 = \text{edim } S_3 + 1$ , we have  $\text{covdim } H = \text{edim } H + 1 \leq 2 \text{edim } S_3 + 1 \leq 3$ . We claim that  $\text{covdim } H = 3$ . Let  $G$  denote  $H \times (\mathbb{Z}/2\mathbb{Z})^2$ . By Corollary 4.4,  $\text{covdim } G = \text{covdim } H + 1$ . Since  $G$  contains a copy of  $(\mathbb{Z}/2\mathbb{Z})^4$ , its covariant dimension is at least 4, hence it is 4, and so the covariant dimension of  $H$  is 3. The same reasoning shows that  $\text{covdim } S_3 \times S_4 = 4$  and  $\text{covdim } S_4 \times S_4 = 5$ .

**Example 5.3.** Let  $G := A_4 \rtimes C_4$  where a generator  $x$  of  $C_4$  acts on  $A_4$  by conjugation with a 4-cycle  $\sigma \in S_4$ . We get

$$Z(G) = \langle x^2 \sigma^2 \rangle \simeq C_2, \quad (G, G) = A_4, \quad G/Z(G) \simeq S_4.$$

Thus  $\text{edim } G = \text{covdim } G = \text{covdim } S_4 = 3$ , by Corollary 3.7. Moreover,  $G$  has a 3-dimensional faithful representation—the standard representation of  $A_4$  lifts to a faithful representation of  $G$ —and  $G$  contains a subgroup isomorphic to  $C_2 \times C_2 \times C_2$ .

**Example 5.4.** Let  $\sigma \in S_n \setminus A_n$  be of (even) order  $m$  where  $n \geq 4$ , and consider the group  $G := A_n \rtimes C_m$  where a generator of  $C_m$  acts on  $A_n$  by conjugation with  $\sigma$ . Again, we can apply Corollary 3.7 and get  $\text{edim } G = \text{covdim } G = \text{covdim } S_n$ .

**Example 5.5.** Let  $G := (C_3 \times C_3) \rtimes (C_4 \times C_8)$  where a generator  $x$  of  $C_4$  acts on  $C_3 \times C_3$  by sending each element to its inverse, and a generator  $y$  of  $C_8$  by sending the first component to its inverse and leaving the second component invariant. Then  $Z(G) = \langle x^2, y^2 \rangle \simeq C_2 \times C_4$ ,  $(G, G) = C_3 \times C_3$  and  $G/Z(G) \simeq S_3 \times S_3$ . Since the center is not cyclic we cannot apply Corollary 3.7 directly, but have to pass through the intermediate group  $\bar{G} := G/\langle x^2 \rangle$  which has a cyclic center, namely  $\langle y^2 \rangle$ . Thus we obtain  $\text{edim } \bar{G} = \text{covdim } \bar{G} = \text{covdim } \bar{G}/Z(\bar{G}) = \text{covdim } S_3 \times S_3 = 3$  by Example 5.2. Since  $\bar{G}$  is faithful we can apply Corollary 4.7: Take  $H := \langle x^2 \rangle$  and choose for  $\chi$  a lift of the character  $\bar{\chi}$  on  $Z(G) = \langle x^2, y^2 \rangle$  given by  $\bar{\chi}(x^2) = -1$  and  $\bar{\chi}(y^2) = 1$ . We finally get  $\text{edim } G = \text{covdim } G = \text{covdim } \bar{G} + 1 = 4$ .

**Example 5.6.** A recent general theorem due to KARPENKO-MERKURJEV [KM08] is the following. *For any finite  $p$ -group  $G$  the essential dimension  $\text{edim } G$  equals the minimal dimension of a faithful representation of  $G$ .* Using this, MEYER-REICHSTEIN [MR08] have found formulas for the essential dimension of all  $p$ -groups. Here we give a simple formula for the essential dimension of semidirect products  $G_p(k, \ell, \alpha) := \mathbb{Z}/p^k \rtimes \mathbb{Z}/p^\ell$  where the generator  $\bar{1}$  of  $\mathbb{Z}/p^k$  induces the automorphism  $\alpha$  on  $A := \mathbb{Z}/p^\ell$ . Our results generalize [KS07, Proposition 6.2].

$$\text{covdim } G_p(k, \ell, \alpha) = \begin{cases} p^k & \text{if } \alpha \text{ has order } p^k, \\ p^d + 1 & \text{if } \alpha \text{ has order } p^d, d < k. \end{cases}$$

Note that  $C := p^{\ell-1}A$  lies in the center of  $G := G_p(k, \ell, \alpha)$ , so that the covariant dimension and essential dimension are the same.

The second case follows from the first using Corollary 4.8. So we assume that  $\alpha$  has order  $p^k$ . Let  $V$  be a faithful  $G$ -module. Then the (cyclic) center of  $G$  acts faithfully on an irreducible component  $W$  of  $V$ , and  $\text{Ker}(G \rightarrow \text{GL}(W))$  is trivial since any nontrivial normal subgroup of  $G$  intersects the center. Thus  $G$  is faithful. (One could also use Proposition 6.1 below.)

Let  $V$  be an irreducible faithful representation of  $G$ . Then, since  $G$  is supersolvable,  $V$  is induced by a character of a proper subgroup  $H$ . We claim that  $H$  is abelian. If not, then  $(H, H) \subset (G, G) \subset A$  contains  $C$ , so  $C$  acts trivially on  $V$ , a contradiction. If  $H$  is an abelian subgroup, we may consider a character of  $H$  which is faithful on  $H$  intersected with the (cyclic) center. Then the induced representation is faithful of dimension  $[G : H]$ . Thus we only need to show that any abelian subgroup  $H$  of  $G$  has order at most  $p^\ell$ .

Let  $\gamma$  generate the canonical projection of  $H$  to  $\mathbb{Z}/p^k$  and let  $y$  generate  $H \cap A$ . We may assume that  $\gamma \neq e$  and that  $y \neq e$ . Now  $H$  is generated by  $x$  and  $y$  where  $x \in H$  has image  $\gamma$  in  $\mathbb{Z}/p^k$ . Choose a generator  $z$  of  $A$  such that  $y = z^{p^r}$  for some  $r$  where  $1 \leq r < l$ . Let  $\gamma(z) = z^{s+1}$ ,  $0 < s < p^l - 1$ . Since  $H$  is abelian, the commutator  $(x, y) = (x, z^{p^r})$  is trivial. It follows that  $\gamma(z)^{p^r} = z^{p^r} = z^{sp^r + p^r}$ , so that  $p^\ell$  divides  $sp^r$  and  $p^{\ell-r}$  divides  $s$ . Hence  $\gamma$  has order at most  $p^r$ . It follows that  $H$  has order at most  $p^\ell$ .

## 6. FAITHFUL GROUPS

Let  $N_G \subset G$  denote the subgroup generated by the minimal subgroups (under set inclusion) among the nontrivial normal abelian subgroups of  $G$ . Our work in [KS07] used the following criterion of Gaschütz.

**Proposition 6.1** ([Gas54]). *Let  $G$  be a finite group. Then  $G$  is faithful if and only if  $N_G$  is generated by the conjugacy class of one of its elements.*

We have the following corollary [KS07, Corollary 4.1], which we need in the next section.

**Corollary 6.2.** *Let  $G$  be a non-faithful group and  $H \subset G$  a subgroup containing  $N_G$ . Then  $H$  is non-faithful, too.*

The proof given in [KS07] claims that  $N_G \subset N_H$ . But this is false. For example, let  $G = S_4 \supset D_4$ . Then  $N_G$  is the Klein 4-group, while  $N_H = Z(D_4) \simeq \mathbb{Z}/2\mathbb{Z}$ . Here is a correct proof.

**Lemma 6.3.** *Let  $N_1, \dots, N_k$  be the minimal nontrivial normal abelian subgroups of a finite group  $G$ . Then*

- (1) *Each  $N_i$  is isomorphic to  $(\mathbb{Z}/p\mathbb{Z})^n$  for some  $n \in \mathbb{N}$  and prime  $p$ .*
- (2) *Let  $L$  be a  $G$ -normal subgroup of  $N_G$ . There is a direct product  $M$  of a subset of  $\{N_1, \dots, N_k\}$  such that  $N_G$  is the direct product  $LM$ .*

*Proof.* By minimality, for any prime  $p$  and  $i$ ,  $pN_i$  is zero or  $N_i$ . Thus  $N_i \simeq (\mathbb{Z}/p\mathbb{Z})^n$  for some  $p$  and  $n$  giving (1). For (2), inductively assume that we have found a  $G$ -normal subgroup  $M_j$  of  $N_G$  which is a direct product of a subset of  $\{N_1, \dots, N_j\}$  such that  $LM_j$  is a direct product containing  $N_1, \dots, N_j$ . We start the induction with  $M_0 = \{e\}$ . If  $LM_j$  contains  $N_{j+1}$ , then set  $M_{j+1} = M_j$ . If not, then  $N_{j+1} \cap LM_j$

must be trivial, so that the products  $M_{j+1} := M_j N_{j+1}$  and  $LM_{j+1}$  are direct where  $N_{j+1} \subset LM_{j+1}$ . Set  $M = M_k$ . Then  $LM$  is a direct product containing all the generators of  $N_G$ , hence equals  $N_G$ .  $\square$

**Corollary 6.4.**  *$N_G$  is a direct product of a subset of  $\{N_1, \dots, N_k\}$ , hence  $N_G$  is abelian.*  $\square$

*Proof of Corollary 6.2.* The subgroup  $N_G \cap N_H \subset N_H$  is normal in  $H$ . By Lemma 6.3 it has a complement  $M$ . Now assume that  $H$  is faithful. Then by Proposition 6.1 there exists an element  $(c, d) \in N_H = (N_G \cap N_H) \times M$  whose  $H$ -conjugacy class generates  $N_H$ . Then the  $H$ -conjugacy class of  $c$  generates  $N_G \cap N_H$ . Now let  $N_i$  be one of the minimal nontrivial normal abelian subgroups of  $G$ . By hypothesis,  $N_i \subset H$ , hence  $N_i$  contains a minimal nontrivial  $H$ -submodule  $N'$ . Then  $N' \subset N_G \cap N_H$ . The smallest  $G$ -stable subspace of  $N_G$  containing  $N'$  is  $N_i$ , hence  $N_i$  lies in the  $G$ -submodule of  $N_G$  generated by the conjugacy class of  $c$ . Since  $N_i$  is arbitrary, we see that  $G$  is faithful.  $\square$

*Remark 6.5.* Let  $G_1, G_2, \dots, G_m$  be faithful groups. Then the product  $G_1 \times \dots \times G_m$  is faithful if and only if the orders of the centers  $Z(G_i)$  are pairwise coprime. In fact, the center of the product is cyclic if and only if the orders  $|Z(G_i)|$  are pairwise coprime, and in this case the tensor product of irreducible faithful representations  $V_i$  of  $G_i$  is irreducible and faithful.

## 7. GROUPS OF COVARIANT DIMENSION 2

In [Led07] it is shown that  $G$  has essential dimension one if and only if admits an embedding into  $\mathrm{GL}_2$  such that the only scalar matrix in the image is the identity. In [KS07] we showed that a finite group of covariant dimension 2 is a subgroup of  $\mathrm{GL}_2$  and thus admits a faithful 2-dimensional representation. In particular, we have the following result (cf. [KS07, Theorem 10.3]).

**Theorem 7.1.** *If  $G$  is a non-faithful finite group of covariant dimension 2, then  $G$  is abelian of rank 2.*

Unfortunately, there is a gap in the proof of Lemma 10.3 in [KS07] which is used in the proof of the theorem. So we give a new proof here which avoids this lemma. We start with the following result.

**Lemma 7.2.** *If  $G$  is a non-commutative finite group of covariant dimension 2, then  $G/Z(G)$  is isomorphic to a subgroup of  $\mathrm{PGL}_2$ .*

*Proof.* We use the notation of section 3. Let  $\varphi: V \rightarrow W$  be a multihomogeneous minimal covariant of degree  $A$ . Set  $X := \overline{\varphi(V)} \subset W$  and let  $S$  denote the image of the homomorphism  $T(A): \mathbb{C}^{*n} \rightarrow \mathbb{C}^{*m}$ . Since  $S$  is non-trivial, Lemma 3.3 shows that  $\dim \mathbb{P}(X) \leq 1$  and that  $G/Z(G)$  acts faithfully on  $\mathbb{P}(X)$ . Thus  $\dim \mathbb{P}(X) = 1$  and  $G/Z(G)$  acts faithfully on the normalization  $\mathbb{P}^1$  of  $\mathbb{P}(X)$ . The lemma follows.  $\square$

*Proof of Theorem 7.1.* Let  $G$  be a minimal counterexample, i.e.,  $G$  is non-faithful and non-commutative of covariant dimension 2, and every strict subgroup is either commutative or faithful. By the lemma above,  $G/Z(G)$  is isomorphic to  $A_5$ ,  $S_4$ ,  $A_4$ , or  $D_{2n}$ , and the image of  $N_G$  in  $G/Z(G)$  is a normal abelian subgroup.

**Claim 1:** *There are no surjective homomorphisms from  $G$  to  $A_5$ ,  $S_4$ , or  $A_4$ .*

If  $\rho$  is a surjective homomorphism from  $G$  to  $A_5$  then  $\rho(N_G)$  is trivial. If  $\rho$  is a surjective homomorphism from  $G$  to  $S_4$  then  $\rho(N_G) \subset K$  where  $K \subset S_4$  is the Klein 4-group. In both cases  $\rho^{-1}(A_4) \subsetneq G$  is neither faithful (by Corollary 6.2) nor commutative, contradicting the minimality assumption.

Now assume that there is a surjective homomorphism  $\rho: G \rightarrow A_4$ , and let  $g_3 \in G$  be the preimage of an element of  $A_4$  of order 3. We may assume that the order of  $g_3$  is a power  $3^\ell$ . Since  $\rho(N_G) \subset K$ , the strict subgroup  $S := \rho^{-1}(K) \subsetneq G$  is commutative. Denote by  $S_2$  the 2-torsion of  $S$ . Since  $\rho(S_2) = K$  we see that  $S_2$  has rank 2. Moreover,  $S_2$  is normalized by  $g_3$ , but not centralized, and so  $\text{covdim}\langle g_3, S_2 \rangle \geq 3$  by [KS07, Corollary 4.4]. This contradiction proves Claim 1.

**Claim 2:** *For every prime  $p > 2$  the  $p$ -Sylow-subgroup  $G_p \subset G$  is normal and commutative of rank  $\leq 2$ . Hence  $G$  is a semidirect product  $G_2 \rtimes G'$  where  $G' := \prod_{p>2} G_p$  and  $G_2$  is a 2-Sylow subgroup.*

From Claim 1 we know that  $G/Z(G) \simeq D_{2n}$ . Then Claim 2 follows, because every  $p$ -Sylow-subgroup of  $D_{2n}$  for  $p \neq 2$  is normal and cyclic.

Now we can finish the proof. The case that  $G = G_2$  is handled in [KS07, Lemma 10.2], so we can assume that  $G'$  is non-trivial. If  $G_2$  commutes with  $G'$ , then  $G_2$  is non-commutative and faithful. Moreover, no  $G_p$  can be of rank 2, else we have a subgroup which is a product  $H := G_2 \times (\mathbb{Z}/p)^2$ , and we have  $\text{covdim} H \geq 3$  by Corollary 4.4. So  $G'$  has rank 1. Then  $G'$  is cyclic, hence  $G$  is faithful by Remark 6.5, which is a contradiction. Hence we may assume that  $G_2$  acts nontrivially on  $G'$ .

It is clear that  $N_G = N_2 \times N'$  where  $N_2 = N_G \cap G_2$  and  $N' := N_G \cap G'$ . Since  $G_2$  acts nontrivially on  $G'$ , there is a  $g \in G_2$  which induces an order 2 automorphism of some  $G_p \neq \{e\}$ . Then one can see that  $g$  acts nontrivially on  $N_{G_p}$ . Since  $G$  is not faithful,  $N_G$  is not generated by a conjugacy class (Proposition 6.1) and the same holds for the subgroup  $H := \langle g, N_2 \rangle \times N'$  (Corollary 6.2). Thus  $H$  is neither faithful nor commutative, so that it must equal  $G$  by minimality. It follows that each nontrivial  $G_p$ , for  $p \neq 2$ , is isomorphic to either  $\mathbb{Z}/p$  or  $(\mathbb{Z}/p)^2$ .

Suppose that  $G_p = (\mathbb{Z}/p)^2$  for some  $p$ . If  $g$  acts trivially on  $G_p$ , then it must act nontrivially on some  $G_q$ , and then we have the subgroup  $(\langle g \rangle \times G_q) \times (\mathbb{Z}/p)^2$  which by Corollary 4.4(2) has covariant dimension at least 3. If  $g$  acts by sending each element of  $G_p$  to its inverse, then, by Corollary 4.4(1) and Corollary 4.5,

$$\text{covdim}\langle g \rangle \times G_p = \text{covdim}(\langle g \rangle \times G_p) \times \mathbb{Z}/p \geq \text{covdim}(\mathbb{Z}/p)^3 = 3.$$

So we can assume that  $g$  acts on  $G_p$  fixing one generator and sending the other to its inverse for every  $G_p$  of rank 2. Thus  $G'$  is generated by the conjugacy class of a single element. It follows that  $N_2$  must have rank 2 and  $g$  must commute with  $N_2$ , else  $N_2 \times G'$  is generated by the conjugacy class of a single element. Suppose that  $\langle g \rangle \cap N_2 \simeq \mathbb{Z}/2$ . If  $g$  acts nontrivially on  $\mathbb{Z}/p \subset G'$ , then  $\langle g, N_2 \rangle \times \mathbb{Z}/p$  contains a subgroup  $(\langle g \rangle \times \mathbb{Z}/p) \times \mathbb{Z}/2$  which has covariant dimension 3, again by Corollary 4.4(2). If  $\langle g \rangle \cap N_2 = \{e\}$ , then we have the subgroup  $(\langle g \rangle \times \mathbb{Z}/p) \times (\mathbb{Z}/2)^2$  which has covariant dimension three by Corollary 4.4(1). This finishes the proof of the theorem.  $\square$

## 8. ERRATA TO [KS07]

First sentence after Definition 4.1. Replace “simple groups.” by “nonabelian simple groups.”

Proof of Proposition 4.3, second paragraph. Replace “is divisible by  $m$ ” with “is congruent to 1 mod  $m$ .”

Proof of Corollary 5.1 last sentence. Replace “Corollary 4.3” by “Proposition 4.3.”

Proof of Proposition 6.1 second paragraph. Change “ $\varphi|_W$ ” to  $F|_W$ .”

Proof of Proposition 6.1 first displayed formula. Replace “ $F(w, t)$ ” and “ $F_0(w, t)$ ” by “ $F(w, t^m)$ ” and “ $F_0(w, t^m)$ .”

Top of page 282. Change “trivial stabilizer” to “trivial stabilizer or stabilizer  $\pm I$ .”

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