

RATIONAL COVARIANTS OF REDUCTIVE GROUPS AND HOMALOIDAL POLYNOMIALS

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ABSTRACT. Let G be a complex reductive group, V a G -module and $f \in \mathcal{O}(V)^G$ a nonconstant homogeneous invariant. We investigate relations between the following properties:

- $df: V \rightarrow V^*$ is dominant,
- f is *homaloidal*, i.e., df induces a birational map $\mathbb{P}(V) \rightarrow \mathbb{P}(V^*)$,
- V is stable, i.e., the generic G -orbit is closed.

If f generates $\mathcal{O}(V)^G$, we show that the properties are equivalent, generalizing results of SATO-KIMURA on prehomogeneous vector spaces.

INTRODUCTION

Our ground field is \mathbb{C} , the field of complex numbers. Throughout this note, G will be a reductive (complex) algebraic group, and V a (finite dimensional algebraic) G -module. We denote by $\mathcal{O}(V)$ the algebra of polynomial functions on V and by $\mathcal{O}(V)^G$ the subalgebra of G -invariant functions. The *algebraic quotient* (of V by G) is the morphism $\pi_V: V \rightarrow V//G$ corresponding to the inclusion $\mathcal{O}(V)^G \subset \mathcal{O}(V)$. Similar definitions and notation apply to an affine G -variety, i.e., an affine variety X equipped with an algebraic action of G .

If $f \in \mathcal{O}(V)^G$, then the differential

$$df: V \rightarrow V^*, \quad v \mapsto df(v): (T_v V \simeq V) \rightarrow \mathbb{C}$$

is G -equivariant, where V^* denotes the dual G -module. The (schematic) fiber $F := f^{-1}(f(v))$ is non-singular at v if and only if $df(v) \neq 0$, in which case $\text{Ker } df(v) = T_v F$, the tangent space to F at v . Note that if f is homogeneous and df is dominant, then df defines a rational map $\Phi_f: \mathbb{P}(V) \rightarrow \mathbb{P}(V^*)$ of finite degree. If Φ_f is birational, then f is said to be *homaloidal*.

We say that V is *stable* if the set of closed G -orbits contains a non-empty open subset of V . The main results of this paper are the following.

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Theorem A. *Let $f \in \mathcal{O}(V)^G$ be a homogeneous invariant. If the differential $df: V \rightarrow V^*$ is dominant, then V is a stable G -module.*

Theorem B. *Assume that the invariant ring $\mathcal{O}(V)^G$ is generated by a homogeneous function f . Then the following are equivalent:*

- (i) *The differential $df: V \rightarrow V^*$ is dominant;*
- (ii) *The differential df induces a birational map $\Phi_f: \mathbb{P}(V) \rightarrow \mathbb{P}(V^*)$, i.e., f is homaloidal;*
- (iii) *The G -module V is stable.*

A crucial step in the proof of Theorem B is Proposition C, below, which is interesting in its own right. In a way it gives conditions for LUNA's Slice Theorem ([Lu73], see also [Sl89]) to hold "generically." Recall that a G -equivariant morphism $\varphi: X \rightarrow Y$ is *excellent* if the induced morphism $\bar{\varphi}: X//G \rightarrow Y//G$ is étale and the following diagram is cartesian:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \downarrow \pi_X & & \downarrow \pi_Y \\ X//G & \xrightarrow{\bar{\varphi}} & Y//G \end{array}$$

In particular, excellent morphisms are étale. A subset S of an affine G -variety X is *saturated* if $S = \pi_X^{-1}(\pi_X(S))$.

Proposition C. *Let X and Y be two smooth affine G -varieties of the same dimension and let $\varphi: X \rightarrow Y$ be a G -equivariant dominant morphism. Assume that the generic fibers (see §1) of π_X and π_Y are G -isomorphic. Then there is a non-empty saturated open set $U \subset Y$ such that the induced morphism $\varphi|_{\varphi^{-1}(U)}: \varphi^{-1}(U) \rightarrow U$ is excellent.*

In the special case of a G -module and its dual the assumption on the generic fibers is automatically fulfilled.

Proposition D. *Let $\varphi: V \rightarrow V^*$ be a G -equivariant dominant morphism. Then the generic fibers of π_V and π_{V^*} are isomorphic.*

The following relations between V and V^* come in handy:

Proposition E. *Let V be a G -module. Then*

- (1) *V is stable if and only if V^* is stable.*
- (2) *The principal isotropy groups (see §1) of V and V^* are the same.*

OUTLINE OF THE PAPER. In §1 we state and prove a generalization (Proposition 1) of Proposition C, and we prove Propositions D and E and Theorem A. In §2 we consider the special case of G -modules with one-dimensional quotient, and we give the proof of Theorem B, which generalizes results obtained by SATO-KIMURA

[SaK77] for prehomogeneous vector spaces. In §3 we show how to obtain the SATO-KIMURA results from Theorem B.

§1. DOMINANT COVARIANTS

Let X be an affine G -variety. Note that any fiber F of $\pi_X: X \rightarrow X//G$ is *without invariants* in the sense that $\mathcal{O}(F)^G = \mathbb{C}$. We say that F is a *generic fiber of π_X* (or of X) if there is a nonempty open subset $Z \subset X//G$ such that $\pi_X^{-1}(Z) \rightarrow Z$ is a G -fiber bundle with fiber F (in the étale topology). LUNA's slice theorem [Lu73] shows that any smooth affine G -variety has a generic fiber F . Moreover, F has the form $G *^H W$ where $H \subset G$ is reductive and W is an H -module without invariants. In particular, F is smooth and G acts transitively on the irreducible components of F . The subgroup H (or any G -conjugate of H) is called a *principal isotropy group of X* . A nonsmooth affine G -variety may not have a generic fiber.

For completeness, we add the following well-known result:

Lemma 1. *Let $\varphi: F \rightarrow F'$ be a dominant G -equivariant morphism where F and F' are G -isomorphic affine varieties without invariants. Then φ is an isomorphism.*

Proof. Since φ is dominant, we have a G -linear inclusion $\varphi^*: \mathcal{O}(F') \hookrightarrow \mathcal{O}(F)$. By assumption, the isotypic components of $\mathcal{O}(F)$ and $\mathcal{O}(F')$ are finite dimensional of the same dimension. Thus φ^* is an isomorphism. \square

The next result is a generalization of Proposition C of the introduction.

Proposition 1. *Let X and Y be irreducible affine G -varieties of the same dimension which have generic fibers F_X and F_Y , respectively. Assume that F_X and F_Y are G -isomorphic and that G acts transitively on the irreducible components of F_X . If $\varphi: X \rightarrow Y$ is a dominant G -equivariant morphism, then there is a non-empty saturated open set $U \subset Y$ such that φ is excellent on $\varphi^{-1}(U)$.*

Proof. It suffices to find a (non-empty) saturated open set $X' \subset X$ such that $\varphi|_{X'}: X' \rightarrow Y$ is excellent, since every such X' contains a dense subset of the form $\varphi^{-1}(U)$ where $U \subset Y$ is open and saturated. Therefore, we can assume that $\pi_X: X \rightarrow X//G$ and $\pi_Y: Y \rightarrow Y//G$ are both fiber bundles over smooth bases with fibers F_X and F_Y , respectively, and that $\bar{\varphi}: X//G \rightarrow Y//G$ is dominant. Shrinking further we can assume that $\bar{\varphi}$ is étale.

Let $X_0 \subset X$ be the (non-empty) G -stable open set where φ is smooth. If a fiber F of π_X meets X_0 , then $\overline{\varphi(F)}$ has the same dimension as F , hence contains an irreducible component of a fiber F' of π_Y . Since G acts transitively on the irreducible components of the fibers, we get that $\overline{\varphi(F)} = F'$, and Lemma 1 shows that φ induces an isomorphism $F \xrightarrow{\sim} F'$.

The argument above shows that we can assume that every fiber of π_X is mapped isomorphically onto a fiber of π_Y . Thus we obtain a map $\tilde{\varphi}: X \rightarrow X//G \times_{Y//G} Y$ of fiber bundles over $X//G$ inducing isomorphisms on the fibers. The following lemma shows that $\tilde{\varphi}$ is an isomorphism, i.e., that φ is excellent. \square

The lemma below is certainly well-known. Since we could not find a suitable reference, we include a short proof.

Lemma 2. *Let $p: X \rightarrow S$ and $q: Y \rightarrow S$ be two fiber bundles with fiber F and let $\varphi: X \rightarrow Y$ be a morphism of fiber bundles (i.e., $q \circ \varphi = p$) inducing isomorphisms $\varphi_s: X_s \xrightarrow{\sim} Y_s$ on the fibers for all $s \in S$. Then φ is an isomorphism.*

Proof. (a) Let us first assume that φ is a finite morphism and let $x \in X$, $y := \varphi(x) \in Y$ and $s := p(x) = q(y)$. Then $\varphi^*: \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$ is finite and injective and induces, by assumption, an isomorphism $\mathcal{O}_{Y,y}/\mathfrak{m}_s \mathcal{O}_{Y,y} \xrightarrow{\sim} \mathcal{O}_{X,x}/\mathfrak{m}_s \mathcal{O}_{X,x}$. It follows that $\mathcal{O}_{X,x} = \mathcal{O}_{Y,y} + \mathfrak{m}_y \mathcal{O}_{X,x}$ and so $\mathcal{O}_{X,x} = \mathcal{O}_{Y,y}$ by NAKAYAMA's Lemma. Hence φ is an isomorphism.

(b) Next we remark that the lemma also holds if Y is normal, since every bijective morphism onto a normal variety is an isomorphism, by ZARISKI's Main Theorem (see [Mu88, Ch. 3, §9]).

(c) In general, let $\tilde{X} \rightarrow X$ be the normalization of X and $\tilde{S} \rightarrow S$ the normalization of S . We obtain a commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\eta_X} & X \\ \downarrow \tilde{p} & & p \downarrow \\ \tilde{S} & \longrightarrow & S \end{array}$$

where all horizontal maps are finite and where \tilde{p} is a fiber bundle with fiber \tilde{F} , the normalization of F . A similar diagram is obtained for Y . Moreover, φ induces a fiber bundle morphism $\tilde{\varphi}: \tilde{X} \rightarrow \tilde{Y}$ which is again an isomorphism on the fibers. It now follows from (b) that $\tilde{\varphi}$ is an isomorphism:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\eta_X} & X \\ \simeq \downarrow \tilde{\varphi} & & \varphi \downarrow \\ \tilde{Y} & \xrightarrow{\eta_Y} & Y \end{array}$$

Therefore, φ is finite and the lemma follows from (a). \square

In order to prove Propositions D and E of the introduction, we use the following well-known lemma:

Lemma 3. *Let G be a connected reductive group. Then there exists an automorphism $\tau: G \xrightarrow{\sim} G$ with the following property: If $\rho: G \rightarrow \mathrm{GL}(V)$ is a representation of G , then $\rho \circ \tau: G \rightarrow \mathrm{GL}(V)$ is equivalent to the dual representation $\rho^*: G \rightarrow \mathrm{GL}(V^*)$.*

As a consequence, we see that for a connected reductive group G any covariant $\varphi: V \rightarrow W$ can be regarded as a covariant $\varphi^*: V^* \rightarrow W^*$ by simply changing the action of G by the automorphism τ . Moreover, the quotient map $\pi_V: V \rightarrow V//G$ can be identified with the quotient map $\pi_{V^*}: V^* \rightarrow V^*//G$.

Remark. It is essential in the lemma above that G be connected. In fact, the smallest Mathieu group M_{11} has a non self-dual representation W of dimension 10, but M_{11} has no outer automorphism. (This example was shown to us by GURALNICK.) We do not know if the invariant rings of W and W^* are isomorphic.

Proof of Proposition D. Let $\varphi: V \rightarrow V^*$ be a dominant covariant. We want to show that the generic fibers of π_V and π_{V^*} are isomorphic. We first assume that G is connected. We have seen above that φ can be regarded as a covariant $\varphi^*: V^* \rightarrow V$, and the composition $\varphi^* \circ \varphi: V \rightarrow V$ is a dominant covariant. It follows from Proposition 1 that for a generic fiber F of V the image $F' := \varphi^*(\varphi(F))$ is also a generic fiber of V , and the composition $F \rightarrow \overline{\varphi(F)} \rightarrow F'$ is an isomorphism. Hence $F \xrightarrow{\sim} \overline{\varphi(F)}$. Since the generic fibers of V and V^* are irreducible and of the same dimension, it follows that $\overline{\varphi(F)}$ is a generic fiber F^* of V^* and that $F \xrightarrow{\sim} F^*$.

In case of a general reductive group G we can apply Propositions C and D to G^0 and find a G^0 -saturated open set $U \subset Y$ such that φ is excellent on $\varphi^{-1}(U)$ with respect to may assume that U is G -saturated. In the following diagram

$$\begin{array}{ccc}
 \varphi^{-1}(U) & \xrightarrow{\varphi} & U \\
 \downarrow \pi_V^0 & & \downarrow \pi_{V^*}^0 \\
 \varphi^{-1}(U) // G^0 & \xrightarrow{\bar{\varphi}^0} & U // G^0 \\
 \downarrow \bar{\pi}_V & & \downarrow \bar{\pi}_{V^*} \\
 \varphi^{-1}(U) // G & \xrightarrow{\bar{\varphi}} & U // G
 \end{array}$$

the upper square is cartesian, the maps π_V^0 and $\pi_{V^*}^0$ are the quotients of the G^0 -action, and the maps $\bar{\pi}_V$ and $\bar{\pi}_{V^*}$ are quotients by the finite group $\bar{G} := G/G^0$. It suffices to show that the generic fibers of $\bar{\pi}_V$ and $\bar{\pi}_{V^*}$ have the same number of elements, or, equivalently, that the generic fibers of π_V and π_{V^*} have the same number of components.

Let G' denote the kernel of the action of G on $V // G^0$. Then the number n_V of components of the generic fiber of π_V equals $[G : G']$. The following lemma shows that $n_V = n_{V^*}$, proving the proposition. \square

Lemma 4. *Let $N \subset G$ be a normal subgroup, and let W be a G -module. Then $(W^N)^*$ and $(W^*)^N$ are isomorphic G/N -modules. In particular, the kernels of the actions of G on W^N and $(W^*)^N$ are the same.*

Proof. One easily reduces to the case that W is an irreducible G -module. Since W^N is G -stable, either N acts trivially on W , or N acts nontrivially on W and $W^N = (0)$. But W^* is also an irreducible G -module, and N acts trivially on W^* if and only if it acts trivially on W . Thus, $(W^N)^*$ and $(W^*)^N$ are either both zero or W^* . \square

Proof of Proposition E. Let V be a G -module, and let V_0 denote V considered as a G^0 -module. Then V is stable if and only if V_0 is stable. Since V_0 is stable if and only if the generic fiber of π_{V_0} is an orbit, the comments following Lemma 3 show that V_0 is stable if and only if V_0^* is stable. Thus stability of V and V^* are equivalent.

Let H be a principal isotropy group of V , and let N denote $N_G(H)/H$. By the Luna-Richardson Theorem [LuR79], the restriction map $\mathcal{O}(V)^G \rightarrow \mathcal{O}(V^H)^N$ is an isomorphism where the action of N on V^H has trivial principal isotropy groups (hence is stable). The action of N on $(V^*)^H \simeq (V^H)^*$ is then also stable with finite principal isotropy groups. By [Lu75], $\mathcal{O}(V^*)^G \rightarrow \mathcal{O}((V^*)^H)^N$ is finite, and every closed N -orbit in $(V^*)^H$ lies in a closed G -orbit in V^* . Since $\mathcal{O}((V^*)^H)^N$ and $\mathcal{O}(V^*)^G$ have the same dimension ($= \dim V // G$), some closed N -orbit in $(V^*)^H$ intersects the closed orbit of a generic fiber of V^* . It follows that $H \subset H^*$, where H^* is a principal isotropy group of V^* . Repeating the argument for V^* shows that H^* is contained in a conjugate of H , hence $H = H^*$. \square

We now give the proof of Theorem A of the introduction.

Proof of Theorem A. Let $\varphi: V \rightarrow V^*$ be a covariant. For any point $v \in V$, the derivative $d\varphi(v): T_v V \rightarrow T_{\varphi(v)} V^* \simeq V^*$ gives a bilinear form β_v on $T_v V \simeq V$, namely, $\beta_v(w_1, w_2) := d\varphi(v)(w_1)(w_2)$. Since φ is a covariant, β_v is G_v -invariant, and β_v is non-degenerate precisely at the points v where φ is étale. Now assume that $f \in \mathcal{O}(V)^G$ and $\varphi := df$. Then the form β_v is the Hessian of f and is symmetric. If φ is dominant, Propositions C and D give us an open G -saturated subset $U \subset V^*$ such that $\varphi|_{\varphi^{-1}(U)}: \varphi^{-1}(U) \rightarrow U$ is excellent, and we may assume that U and $\varphi^{-1}(U)$ are affine. The following result of LUNA (see [Lu72], [Lu73]) shows that the action of G on $\varphi^{-1}(U)$ is stable, proving Theorem A. \square

Theorem (Luna). *Let X be a smooth affine G -variety. Assume that for each $x \in X$ there is a G_x -invariant non-degenerate symmetric bilinear form on the tangent space $T_x X$. Then X is stable.*

§2. ONE-DIMENSIONAL QUOTIENTS

Let V be a G -module such that the invariant ring $\mathcal{O}(V)^G$ is generated by the homogeneous function f of degree d . Then $\mathcal{O}(V^*)^G$ is also generated by a homogeneous function f^* of degree d .

Proposition 2. *Let V , G , f and f^* be as above and let $Gv = \overline{Gv}$ be a closed orbit in V . Then*

- (1) $f^* \circ df$ is a non-zero multiple of f^{d-1} .
- (2) $df^*(df(v)) = c_v v$ for some non-zero $c_v \in \mathbb{C}$.
- (3) If V is stable, then df is dominant.

Proof. We may choose an isomorphism $V \simeq \mathbb{C}^n$ such that $G \subset \mathrm{GL}_n(\mathbb{C})$ is the complexification of a compact subgroup $K \subset \mathrm{U}_n(\mathbb{C})$. Let $(,)$ denote the standard

hermitian inner product on \mathbb{C}^n . Then Kempf-Ness theory ([KeN79], cf. [Sch89]) tells us the following: Let $v \in V$. Then

$$(*) \quad T_v(Gv) \perp v \text{ if and only if } Gv \text{ is a closed orbit and } (v, v) = \inf_{g \in G} (gv, gv).$$

Assume that V is stable, and choose $0 \neq v \in V$ such that Gv is closed. We may assume that (gv, gv) achieves its minimum at v . If $df(v) = 0$, then df vanishes on $G(\mathbb{C}v)$ which is dense in V . Thus $df \equiv 0$, which is impossible. Let λ denote the linear form $df(v) \neq 0$. Then $T_v(Gv) = \text{Ker } \lambda$ has dimension $n - 1$, and so $\lambda(w) = c(w, v)$ for all $w \in V$ and a suitable $c \in \mathbb{C}$. Considered as an element of V^{**} , v lies in the annihilator of $T_\lambda(G\lambda)$: if $A \in \mathfrak{g}$, then $(A\lambda)(v) = \lambda(-Av) = c(-Av, v) \in (\mathfrak{g}v, v) = \{0\}$. But this shows that $\lambda \perp T_\lambda(G\lambda)$ (using the induced hermitian inner product on V^*). Thus $G\lambda$ is a closed orbit in V^* and $\lambda \in G\lambda$ is a point of minimal norm. Hence $f^*(\lambda) \neq 0$, giving (1). Since V^* is also stable, we may apply the reasoning above to $v' := df^*(\lambda) \in V^{**}$. We obtain that v' generates the annihilator of $T_\lambda(G\lambda)$, so that v' is a non-zero multiple of $v \in V^{**}$. We have (2), and (3) follows since $G(\mathbb{C}\lambda)$ is dense in V^* .

In general, let H be a principal isotropy group of V . Then the action of $N := N_G(H)/H$ on V^H is stable, and by the Luna-Richardson Theorem [LuR79], the restriction \bar{f} of f to V^H generates the invariant ring $\mathcal{O}(V^H)^N$. For $v \in V^H$, we have $df(v) \in (V^*)^H$, and using the canonical isomorphism $(V^*)^H \simeq (V^H)^*$ this linear function identifies with $df(v)|_{V^H} = d\bar{f}(v)$. Proposition E shows that H is also the principal isotropy group of V^* . Hence $\bar{f}^* = f^*|_{V^{*H}}$ generates the invariant ring $\mathcal{O}(V^{*H})^N$. By the first part of our proof, $f^*(df(v)) = \bar{f}^*(d\bar{f}(v)) \neq 0$ and $df^*(df(v)) = d\bar{f}^*(d\bar{f}(v))$ is a multiple of v , so we have (1) and (2). \square

Corollary 1. *Let V , f and f^* be as above. If $df: V \rightarrow V^*$ is dominant, then $df^* \circ df = cf^{d-2} \text{Id}_V$ for a non-zero constant $c \in \mathbb{C}$.*

Proof. Since df is dominant, V is stable by Proposition 1 and $df^*(df(v))$ is a nonzero multiple of v for every v not in the nullcone $\mathcal{N}_V = f^{-1}(0)$ of V . Since $G \times \mathbb{C}^*$ has a dense orbit in V and since $df^* \circ df$ is G -equivariant and homogeneous, $df^* \circ df(w) = h(w)w$ for all $w \in V$, where $h(w)$ is a nonzero homogeneous polynomial of degree $d(d-2)$. Clearly, h is G -invariant, hence it has the given form. \square

We now give the proof of Theorem B of the introduction.

Proof of Theorem B. If df induces a birational map $\Phi_f: \mathbb{P}(V) \rightarrow \mathbb{P}(V^*)$, then df is dominant, and conversely by Corollary 1. Dominance of df and stability of V are equivalent by Proposition 2 and Theorem A. \square

Examples. It is easy to find cases where df is not dominant. Start with any G -module V with a one-dimensional quotient $f: V \rightarrow \mathbb{C}$, let H be a reductive group and W a nonzero H -module without invariants. Then $\tilde{V} := V \oplus W$ is a $\tilde{G} := G \times H$ -module with a one-dimensional quotient given by $\tilde{f}: (v, w) \mapsto f(v)$. Clearly $d\tilde{f}(\tilde{V}) \subset V$, so df is not dominant.

§3. PREHOMOGENEOUS VECTOR SPACES

Let V be a prehomogeneous G -module and χ a character of G . A function $f: V \rightarrow \mathbb{C}$ is called a *semi-invariant (with character χ)* if $f(gv) = \chi(g)f(v)$ for all $g \in G$. It is easy to see that semi-invariants have to be homogeneous functions. Note that the subalgebra $\mathbb{C}[f] \subset \mathcal{O}(V)$ is the invariant ring under $G_0 := \text{Ker } \chi$. In fact, f is G_0 -invariant and the fibers of the restriction of f to the dense orbit Gv are exactly the G_0 -orbits.

We say that V is *regular* if there is a semi-invariant f whose Hessian does not vanish identically; equivalently (by Theorem B), f is homaloidal. The following result is essentially contained in the work [SaK77] of SATO-KIMURA.

Theorem 3. *Let V be prehomogeneous, and let $v \in V$ such that Gv is dense in V . Then the following are equivalent:*

- (1) V is regular.
- (2) There is a semi-invariant f with character χ such that V is a stable G_0 -module, where $G_0 := \text{Ker } \chi$ (a reductive group).
- (3) The isotropy group G_v is reductive.
- (4) The complement of Gv is a hypersurface.

If these conditions hold then a semi-invariant f is homaloidal if and only if its zero set is the complement of the dense orbit Gv .

Proof. Given a semi-invariant f , let χ_f denote the corresponding character. If V is regular, then we have a semi-invariant f such that $df: V \rightarrow V^*$ is dominant, and an application of Theorem A (or B) shows that V is a stable ($G_0 := \text{Ker } \chi_f$)-module. Thus (1) implies (2). If G_0 and f are as in (2), then the isotropy group $(G_0)_v$ is reductive. Since $f(v) \neq 0$, we have $G_v = (G_0)_v$, giving (3). If (3), then $Gv \simeq G/G_v$ is affine, which forces the complement $V \setminus Gv$ to be a hypersurface, giving (4). If $S := V \setminus Gv$ is a hypersurface, let f be a function defining S . Then f is a homogeneous semi-invariant, and the fibers of $f: V \rightarrow \mathbb{C}$ over $\mathbb{C} \setminus \{0\}$ are the ($G_0 := \text{Ker } \chi_f$)-orbits in Gv . Since all of these orbits have dimension $\dim V - 1$, they are all closed. Hence the action of G_0 on V is stable, and V is regular by Theorem B. Thus (4) implies (1).

Let f be a semi-invariant. If $f^{-1}(0) = S := V \setminus Gv$, we have shown that f is homaloidal. Suppose that $f^{-1}(0) \neq S$. Then there is an irreducible component S' of S such that $f|_{S'}: S' \rightarrow \mathbb{C}$ is dominant. It follows that the generic fiber of f consists of a ($G_0 := \text{Ker } \chi_f$)-orbit in Gv , together with at least one other orbit. Hence V is not a stable G_0 -module, and f is not homaloidal. \square

Remark. The last statement of the theorem above proves a very special case of a conjecture by DOLGACHEV claiming that a homogeneous polynomial f with prime decomposition $\prod_i f_i^{n_i}$ is homaloidal if and only if $\prod_i f_i$ is homaloidal (see [Do00], end of section 3).

Example. Let $G = \text{SL}_2 \times (\mathbb{C}^*)^3$ and $V = (\mathbb{C}^2)^3$. The group SL_2 acts on each copy of \mathbb{C}^2 in the standard way, and $(t_1, t_2, t_3) \in (\mathbb{C}^*)^3$ sends $(v_1, v_2, v_3) \in V$

to (t_1v_1, t_2v_2, t_3v_3) . Clearly V is a prehomogeneous G -module. Let $f_{ij}(v_1, v_2, v_3)$ denote the determinant $\langle v_i, v_j \rangle$, $1 \leq i < j \leq 3$. Then the semi-invariants are the products of the f_{ij} , and a semi-invariant is homaloidal if and only if it has each of f_{12} , f_{13} and f_{23} as a factor.

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