ON THE SEPARATION PROPERTY OF ORBITS IN REPRESENTATION SPACES

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January 2002

Abstract. A subset X of a vector space V is said to have the "Separation Property" if it separates linear forms in the following sense: Given a pair (α, β) of linearly independent linear forms on V there is a point x on X such that $\alpha(x) = 0$ and $\beta(x)$ is non-zero. A more geometric way to express this is the following: Every linear subspace $H \subset V$ of codimension 1 is linearly spanned by its intersection with X.

The separation property was first asked for conjugacy classes in simple Lie algebras, in connection with some classification problems. We give a general answer for orbits in representation spaces of algebraic groups and discuss in detail some special cases. We also introduce a *strong* and a *weak* separation property which come up very naturally in our setting. It turns out that these separation properties have a number of very nice features. For example, we discovered the surprising fact that in an irreducible representation of a connected semisimple group every linear hyperplane meets every orbit, and we show that a generic orbit always has the separation property.

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Typeset by $\mathcal{AMS}$-TEX

§1. INTRODUCTION

The starting point of this paper was the following question asked by JENS-CARSTEN JANTZEN in relation with some work of ALEXANDER PREMET [Pre97]:

Question. *Let K be an algebraically closed field. Is it true that the minimal nilpotent conjugacy class $C_{\min}$ in the Lie algebra $\mathfrak{g}$ of a simple affine algebraic K-group has the following "Separation Property"?*

(SP) *For any pair of linearly independent linear functions α, β on $\mathfrak{g}$ there is an element $x \in C_{\min}$ such that $\alpha(x) = 0$ and $\beta(x) \neq 0$.*

We will show that the answer is "yes" except for $\mathfrak{sl}_2$ and the symplectic Lie algebras $\mathfrak{sp}_{2n}$ ($n \geq 2$) where we give explicit counterexamples (see §4, Corollary 1 and Example 5). As a consequence, some results in the paper [Pre97] of PREMET do not hold in the given generality. (In the mean time we have been informed that the basic results of that paper are still correct and are now proved along different lines (see [Jan98]).) Nevertheless, the separation property is an interesting concept in itself that we feel is worth further study.

In §2 we first discuss the case of matrices and show that the set of nilpotent rank 1 matrices in $\mathfrak{sl}_n(K) \subset M_n(K)$ has the separation property (SP) in case $n \geq 3$ (Proposition 1).

In §3 we discuss the separation property in general and introduce in addition the "strong" and the "weak" separation property, (SSP) and (WSP). A number of examples illuminate these concepts.

In §4 and §5 we study the separation property for orbits in representation spaces of algebraic groups and give a classification of those representations of reductive groups where the minimal orbit $O_{\min}$ satisfies the various separation properties. The main results are the following.

Theorem 1. *Let G be a connected semisimple group and let $\lambda = \sum_i n_i \omega_i$ be a dominant integral weight expressed in terms of fundamental weights. Let $V(\lambda)$ be the corresponding Weyl module and denote by $O_{\min} \subset V(\lambda)$ the orbit of the highest weight vector.*

1. $\overline{O_{\min}}$ *satisfies* (SSP) $\iff \lambda$ *is a fundamental weight.*
2. $O_{\min}$ *satisfies* (SP) $\iff n_i \leq 1$ *for all i.*
3. $O_{\min}$ *satisfies* (WSP) $\iff n_i \leq 2$ *for all i.*

Theorem 2. *Let V be an irreducible representation of a connected semisimple group G. Assume that the separation property holds for the minimal orbit $O_{\min}$ in V. Then the separation property holds for any G-orbit O. In particular, every hyperplane meets the orbit O.*

An interesting result used in this context is the following.

Lemma 3. *Let G be a connected algebraic group and $\rho\colon G \to \mathrm{GL}(V)$ a representation which does not admit 1-dimensional subquotients. Let $H \subset V$ be a linear subspace of codimension 1.*

(a) *H meets every non-empty G-stable subset X of V, and $H \cap X \neq X$ for $X \neq \{0\}$.*

(b) *If $X \subset V$ is a constructible G-stable subset then $\overline{H \cap X} = H \cap \overline{X}$.*

We also show that the generic orbit in an irreducible representation of a semisimple group has the separation property, at least in characteristic zero.

Theorem 3. *Assume* $\operatorname{char} K = 0$. *Let $O \subset V$ be a generic orbit of an irreducible representation of a semisimple group G. Then O has the separation property* (SP). *More precisely, the intersection $H \cap O$ with every hyperplane is reduced.*

In §6 we discuss the case of the group SL_2 in characteristic zero and its representations on binary forms. These results are strongly related to the work of KARIN BAUR [Bau01].

Theorem 4. *Let V_n denote the binary forms of degree n (> 1), considered as a representation of* SL_2. *If the form $f \in V_n$ contains a linear factor of multiplicity one, then the orbit $O_f \subset V_n$ has the separation property.*

This is in a sense the best possible result one can get since we also show that the orbit of x^2y^{n-2} does not have the separation property for $n \geq 4$ (§6, Remark 7).

Finally, in §7 we prove that the strong separation property (SSP) is an open property.

Proposition 5. *Let $p\colon F \to S$ be a family if d-dimensional closed subvarieties of $\mathbb{P}^n$. Then the subset $\{s \in S \mid p^{-1}(s)$ satiesfies* (SSP)$\}$ *is open in S.*

Acknowledgement. We would like to thank Jochen Kuttler for helpful discussions, and Vikram Mehta and Jens-Carsten Jantzen for their help with questions in positive characteristic. Our thanks also go to the referee who pointed out a number of inaccuracies in our setup in a previous version of the paper and made several useful suggestions to improve the paper.

§2. TRACE ZERO MATRICES

Let K be an infinite field and $M_n = M_n(K)$ the algebra of $n \times n$-matrices with entries in K. Denote by $M_n^0 \subset M_n$ the subspace of matrices with trace zero and define

$$C_{\min} := \{X \in M_n^0 \mid X \text{ has rank } 1\}.$$

This is the "smallest" nilpotent conjugacy class different from $\{0\}$: It is contained in the closure of every non-zero nilpotent conjugacy class. (Recall that a matrix of rank one and trace zero is nilpotent.)

Proposition 1. *If $n \geq 3$ then the minimal conjugacy class $C_{\min} \subset M_n^0$ has the following "Separation Property":*

(SP) *For any pair of linearly independent linear functions α, β on M_n^0 there is a matrix $X \in C_{\min}$ such that $\alpha(X) = 0$ and $\beta(X) = 1$.*

Proof. Let $p \geq 0$ denote the characteristic of K. Put $V := K^n$ and so $M_n = \operatorname{End} V$. If we identify $\operatorname{End} V$ with $V^* \otimes V$ in the usual way then the matrices of $C_{\min}$ are of the form $X = \lambda \otimes v$ where $v \in V$, $\lambda \in V^*$ and $\lambda(v) = 0$.

A linear function α on M_n^0 has the form $\alpha(X) = \operatorname{tr}(AX)$ with a suitable matrix $A \in M_n$. If p does not divide n then A can be chosen in M_n^0 and is uniquely determined by α. If p divides n then A is only determined up to a multiple of the identity matrix E_n. For $X = \lambda \otimes v$ we have $\operatorname{tr}(AX) = \lambda(Av)$. With these notation consider the following statement:

$$\text{(1)} \qquad \textit{For all } X \in C_{\min} \textit{ we have: } \quad \alpha(X) = 0 \;\Rightarrow\; \beta(X) = 0.$$

If α, β correspond to $A, B \in M_n$ as above, then this statement is equivalent to the following:

$$\text{(2)} \qquad \textit{For all } v \in V \textit{ and all } \lambda \in V^* \textit{ we have: } \quad \lambda(v) = \lambda(Av) = 0 \;\Rightarrow\; \lambda(Bv) = 0.$$

Hence, the separation property for $C_{\min} \subset M_n^0$ can be formulated in the following way:

(SP') *For given matrices $A, B \in M_n$ assume that $Bv \in KAv + Kv$ for all $v \in V$. Then B is a linear combination of A and E.*

In fact, assume that $B = sA + tE$. If p does not divide n then $B = sA$ since A and B have trace zero. If p divides n then either $B = tE$ or we can replace A by $A' := A + \frac{t}{s}E$ and get $B = sA'$. In all cases it follows, that the linear form β is a multiple of α, hence we get (SP).

In order to prove the statement (SP'), we can replace the field K by any field extension and therefore assume that K is algebraically closed. Consider the family $L_{s,t} := B + sA + tE \in M_n$ where $(s,t) \in K^2$. The assumption in (SP') means that

$$(*) \qquad \bigcup_{s,t \in K} \ker L_{s,t} = V.$$

We have to show that this implies that $L_{s,t} = 0$ for a suitable $(s,t) \in K^2$. So let us assume that $L_{s,t} \neq 0$ for all (s,t).

For a given s the matrix $L_{s,t}$ is invertible for almost all t. Thus

$$C := \{(s,t) \in \mathbb{C} \mid \det L_{s,t} = 0\}$$

is a (possibly reducible) curve in K^2 and $\bigcup_{s,t \in K} \ker L_{s,t} = \bigcup_{(s,t) \in C} \ker L_{s,t}$.

We first remark that the condition $(*)$ implies that there are infinitely many (s,t) such that $\operatorname{rk} L_{s,t} = 1$. Otherwise, we have $\dim(\ker L_{s,t}) \leq n-2$ for all but finitely many (s,t). But this implies that

$$\dim \overline{\bigcup_{s,t\in K} \ker L_{s,t}} = \dim \overline{\bigcup_{(s,t)\in C} \ker L_{s,t}} \leq \dim C + (n-2) = n-1$$

and so $(*)$ cannot hold.

If $L_{s,t}$ has rank 1 then $B + sA$ has an eigenspace of dimension $n-1$, namely the kernel of $L_{s,t}$. It follows that there are infinitely many s such that $B + sA$ has an eigenspace of dimension $n-1$. Hence there is a Zariski-dense set $Z \subset K^2$ such that the matrices $s'B + sA$ have an eigenspace of dimension $n-1$ for all $(s',s) \in Z$.

Claim: The set Y of all matrices $C \in M_n$ having an eigenspace of dimension $\geq n-1$ is closed.

It follows that all the matrices $s'B+sA$ have an eigenspace of dimension $\geq n-1$. In particular, A has an eigenspace U of dimension $n-1$ since we can assume that A is not a multiple of the identity matrix. The assumption $Bv \in KAv + Kv$ for all $v \in V$ from (SP') implies that U is also an eigenspace of B and therefore a subspace of an eigenspace of every linear combination $L_{s,t} := A + sB + tE$. It follows that the kernel of every $L_{s,t}$ is either U or of dimension 1. Since $\dim V > 2$ this contradicts the assumption $(*)$.

It remains to prove the claim above. Consider the following subspace of block matrices

$$M := \left\{ \begin{bmatrix} aE_{n-1} & b \\ 0 & c \end{bmatrix} \right\} \subset M_n.$$

Clearly, $M \subset Y$. It follows from the Jordan normal form that every matrix with an eigenspace of dimension $\geq n-1$ is conjugate to a matrix in M. Thus $Y = \bigcup_{g\in \mathrm{GL}_n} gMg^{-1}$. Since M is stable under conjugation with the Borel subgroup B of upper triangular matrices it follows that Y is closed. (For another proof see [NeS].) □

Remark 1. It is easy to see that the separation property (SP) does not hold for M_2^0: Define $\alpha(\left[\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right]) := b$ and $\beta(\left[\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right]) := a$. Since $C_{\min} \subset M_2^0$ is given by the equation $a^2 + bc = 0$ we see that $\alpha(X) = 0$ implies that $\beta(X) = 0$ for all $X \in C_{\min}$.

We leave it to the reader to show that the separation property (SP) does hold for the set $C \subset M_2$ of all rank 1 matrices in the space of all 2×2 matrices.

§3. SEPARATION PROPERTIES

Let K be an algebraically closed field of characteristic $p \geq 0$ and let V be a finite dimensional K-vector space.

Definition 1a. A subset $X \subset V$ has the *separation property* (SP) if for any two linearly independent linear functions α and β on V there is a point $x \in X$ such that $\alpha(x) = 0$ and $\beta(x) \neq 0$.

The separation property (SP) for $X \subset V$ means that for any pair $H \neq H'$ of linear hyperplanes in V we have $H \cap X \nsubseteq H'$. Or, equivalently, *for any linear hyperplane H, the intersection $H \cap X$ linearly spans H.*

Clearly, we have a similar definition for subsets of projective spaces:

Definition 1b. A subset $Y \subset \mathbb{P}(V)$ has the *separation property* (SP) if for any pair $D \neq D'$ of linear hyperplanes in $\mathbb{P}(V)$ we have $D \cap X \nsubseteq D'$.

Example 1. (a) Let $C \subset \mathbb{P}^2$ be an (irreducible) plane curve of degree d. Then C does not have the separation property (SP) for $d = 2$ and 3. However, a generic curve of degree $d > 3$ has the separation property (SP).
In fact, a plane curve C fails to have the separation property if and only if there is a line L which meets C in a single point. Such a line always exists if $d \leq 3$. In general, the existence of such a line is a closed condition on the curves of a fixed degree d defining a closed subset of $K[x, y, z]_d$ of codimension $\geq d - 3$.

(b) The determinantal variety D_r consisting of all matrices of rank $\leq r$ in $\mathbb{P}(M_{n\times m})$ has the separation property (SP) for all $r \geq 1$ (see §4, Theorem 1: D_1 is the minimal orbit in $M_{n\times m}$, considered as a representation of $\mathrm{SL}_n \times \mathrm{SL}_m$).

(c) If $Y \subset \mathbb{P}^n$ is a smooth complete intersection of dimension ≥ 3 not contained in a linear subspace, then Y has the separation property (SP). (See Proposition 2 for a more general statement.)
In fact, under these assumptions we have $\operatorname{Pic} Y = \mathbb{Z}$ and the Picard group is generated by a hyperplane section, see [Har70, p. 179, Corollary 3.2]. Therefore, every hyperplane section $Y \cap L$ is irreducible and reduced. Now the claim follows from Lemma 1 below.

Let X be an irreducible variety and assume that X is smooth in codimension 1, i.e., that the singular locus has at least codimension 2. Then we can talk about the group $\operatorname{Div} X$ of Weil divisors and the divisor class group $\operatorname{Cl} X$ (see [Har77, II.6]). Let $K(X)$ be the field of rational functions on X. If $f \in K(X)$ is a non-zero rational function then $(f) = (f)_X \in \operatorname{Div} X$ denotes the *principal divisor* defined by f.

Lemma 1. *Let X be an irreducible variety which is smooth in codimension 1, and let $\eta\colon \tilde{X} \to X$ be the normalization. Then η induces a natural isomorphism*

$$\eta^*\colon \operatorname{Div} X \xrightarrow{\sim} \operatorname{Div} \tilde{X}$$

such that for $f \in K(X) = K(\tilde{X})$ we have $\eta^*((f)_X) = (f)_{\tilde{X}}$. Therefore, we also get an isomorphism $\eta^*\colon \operatorname{Cl} X \xrightarrow{\sim} \operatorname{Cl} \tilde{X}$.

Let $f \in K(X)$ be a non-zero rational function on X.

1. *If $(f)_X \geq 0$ then $f \in \mathcal{O}(\tilde{X})$.*
2. *If X is projective or a closed affine cone and if $(f)_X = 0$, then $f \in K$.*

Proof. If $D \subset X$ is an irreducible divisor then $\eta^{-1}(D)$ has a unique irreducible component $\tilde{D}$ of codimension 1 in $\tilde{X}$. We set $\eta^*(D) := \tilde{D}$. This defines an isomorphism $\eta^*\colon \operatorname{Div} X \to \operatorname{Div} \tilde{X}$. Moreover, the canonical homomorphism $\mathcal{O}_{D,X} \to \mathcal{O}_{\tilde{D},\tilde{X}}$ is an isomorphism, by assumption. (As usual, $\mathcal{O}_{D,X}$ denotes the discrete valuation ring of $K(X)$ consisting of those rational functions which are defined on a non-empty open set of D.) Thus $\eta^*((f)_X) = (f)_{\tilde{X}}$ for every non-zero $f \in K(X) = K(\tilde{X})$, and so we get an isomorphism $\eta^*\colon \operatorname{Cl} X \xrightarrow{\sim} \operatorname{Cl} \tilde{X}$.

(1) If $(f)_X \geq 0$ then $f \in \mathcal{O}_{D,X}$ for all irreducible hypersurfaces $D \subset X$ and so $f \in \bigcap_D \mathcal{O}_{D,X}$. We claim that this intersection equals $\mathcal{O}(\tilde{X})$. If X is affine this follows from [Har77, Proposition II.6.3A], and the general case is an immediate consequence.

(2) The assumption $(f)_X = 0$ implies, by (1), that f and f^{-1} belong to $\mathcal{O}(\tilde{X})^*$. If X is projective then so is $\tilde{X}$ and thus $\mathcal{O}(\tilde{X}) = K$. If X is a closed affine cone then $\mathcal{O}(X)$ is (positively) graded with $\mathcal{O}(X)_0 = K$. If follows that $\mathcal{O}(\tilde{X})$ is also graded and that $\mathcal{O}(\tilde{X})_0 = K$. Therefore, in both cases the invertible functions on $\tilde{X}$ are constant. □

We now assume that $Y \subset \mathbb{P}^n$ is a closed subvariety, smooth in codimension 1 (e.g., Y is normal). For any irreducible hypersurface $F \subset \mathbb{P}^n$ not containing Y we denote by $F \cdot Y$ the (effective) divisor on Y defined by F. This means the following: F is defined by an irreducible homogeneous function f of degree d. For any irreducible component D_i of $F \cap Y$ we choose a linear function ℓ not vanishing on D_i and define $n_i \in \mathbb{N}$ to be the order of $\dfrac{f}{\ell^d}$ in the valuation ring $\mathcal{O}_{D_i,Y}$. Then $F \cdot Y := \sum_i n_i D_i$. Clearly, this definition extends to any divisor $F = \sum_j d_j F_j$ of $\mathbb{P}^n$ whose components F_j do not contain Y. Also, if h is a rational function on $\mathbb{P}^n$ defining a rational function $\bar{h}$ on Y, then $(h)_{\mathbb{P}^n} \cdot Y = (\bar{h})_Y$. (For all this and the following see [Har77, II.6].)

If we have $F_1 \cdot Y = F_2 \cdot Y$ for two irreducible hypersurfaces F_1, F_2 not containing Y, then $F_1 = F_2$. In fact, if f_1, f_2 are the corresponding irreducible homogeneous functions, then $\dfrac{f_1}{f_2}$ defines a rational function on Y whose divisor is zero, by assumption. Thus f_1 is a scalar multiple of f_2 by Lemma 1(2) above.

Now let D, D' be two linear hyperplanes in $\mathbb{P}^n$ not containing Y. In general, the condition $D \cap Y = D' \cap Y$ is not sufficient to conclude $D = D'$. However, if $D \cap Y = D' \cap Y$ and, in addition, $D \cap Y$ is irreducible or the divisor $D \cdot Y$ is reduced, then we have $D = D'$. Thus, we have the following result.

Lemma 2. *Assume that $Y \subset \mathbb{P}^n$ is a closed subvariety, smooth in codimension 1 and not contained in a linear subspace of $\mathbb{P}^n$.*

1. *If every intersection $D \cap Y$ with a linear hyperplane $D \subset \mathbb{P}^n$ is generically reduced (i.e. reduced in a dense subset), then Y has the separation property* (SP)*.*

(2) *If every intersection $D \cap Y$ with a linear hyperplane $D \subset \mathbb{P}^n$ is irreducible, then $L \cap Y$ has codimension 2 in Y for every linear subspace $L \subset \mathbb{P}^n$ of codimension 2. In particular, Y has the strong separation property* (SSP) *in the sense of the following definition.*

(Similar statements hold for closed subsets and closed cones in vector spaces.)

Definition 2. A closed subvariety $Y \subset \mathbb{P}(V)$ of dimension ≥ 2 has the *strong separation property* (SSP) if for any linear subspace $L \subset \mathbb{P}(V)$ of codimension 2 we have $\operatorname{codim}_Y L \cap Y = 2$.
The same definition applies to closed cones $X \subset V$: X has the *strong separation property* (SSP) if for any linear subspace $W \subset V$ of codimension 2 we have $\operatorname{codim}_X W \cap X = 2$.

It is clear that the strong separation property (SSP) implies the separation property (SP). On the other hand, the determinantal variety $D_1 \subset M_2(K)$ given by $\det = 0$ has the separation property, but does not satisfy (SSP), since D_1 contains linear planes.

Let us finally introduce the "weak" separation property.

Definition 3. A subset $X \subset V$ has the *weak separation property* (WSP) if for any two linear hyperplanes $H_1 \neq H_2$ we have $H_1 \cap X \neq H_2 \cap X$.
Similarly we define the weak separation property for subsets $Y \subset \mathbb{P}(V)$.

Obviously, the separation property (SP) implies the weak separation property (WSP). On the other hand, the nullcone of $M_2(K)$ consisting of all nilpotent matrices satisfies (WSP), but not (SP).

Example 2. Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree > 1. If $n \geq 3$ then X satisfies (SP) (see Example 1(c) above for a more general statement). If $n \geq 4$ then X satisfies (SSP).

Proof. Let $Y \subset V = K^{n+1}$ be the cone over X and let f be the irreducible homogeneous polynomial of degree > 1 defining Y. Consider the homogeneous map $df : V \to V^*$ and denote by $F := df|_Y : Y \to V^*$ its restriction to Y. By assumption, $F^{-1}(0) = \{0\}$ and so F is a finite morphism.

Let $\alpha \in V^*$, $\alpha \neq 0$, and let $L_\alpha \subset V$ denote the subspace defined by $\alpha = 0$. Then $F^{-1}(K\alpha) = (Y \cap L_\alpha)_{\text{sing}}$, the singular points of the schematic intersection $Y \cap L_\alpha$. In fact, if $v \in Y \cap L_\alpha$ is a singular point, then $d\alpha_v (= \alpha)$ is a multiple of df_v, i.e. $F(v) \in K\alpha$. Conversely, $df_v = \alpha$ for some $v \in Y$ implies that $\alpha(v) = df_v(v) = 0$ and so $v \in Y \cap L_\alpha$, and this is a singular point since $d\alpha_v = \alpha$ is a multiple of df_v.

For $n \geq 3$ we have $\dim Y \cap L_\alpha \geq 2$ and so the singular points of the schematic complete intersection $Y \cap L_\alpha$ have codimension at least 1. Hence $Y \cap L_\alpha$ is reduced and the claim follows from Lemma 2. If $n \geq 4$ then the singularities have at least codimension 2. Therefore, $Y \cap L$ is normal and hence irreducible, and the claim follows also from Lemma 2. $\square$

Example 3. Let $f, g \in K[x_0, \dots, x_n]$ be two homogeneous polynomials of degree > 1 and assume that $X := \mathcal{V}(f, g) \subset \mathbb{P}^n$ is of codimension 2. If $\dim X_{\text{sing}} < n - 5$ then X has the strong separation property (SSP).

Proof. Let $Y \subset V := K^{n+1}$ be the cone over X. By assumption, Y is a complete intersection of dimension $n-1$ with $\dim Y_{\text{sing}} \leq n-4$. If Y does not satisfy (SSP) then there is a linear subspace $E \subset V$ of codimension 2 such that $E \cap Y$ has codimension ≤ 1 in E. If follows that either E is contained in $\mathcal{V}(f)$ or $\mathcal{V}(g)$ or the two hypersurfaces $E \cap \mathcal{V}(f)$ and $E \cap \mathcal{V}(g)$ have a common irreducible component. By a change of coordinates this means that there are homogeneous polynomials $h, f_0, g_0 \in K[x_0, \dots, x_{n-2}]$ and $f_1, f_2, g_1, g_2 \in K[x_0, \dots, x_n]$ such that $\deg h > 0$ and

$$f = hf_0 + x_{n-1}f_1 + x_n f_2 \quad \text{and} \quad g = hg_0 + x_{n-1}g_1 + x_n g_2$$

An easy calculation shows that the closed subset

$$S := \{a = (a_0, \dots, a_{n-2}, 0, 0) \mid h(a) = 0, f_i(a) = g_i(a) \text{ for } i = 1, 2, 3\} \subset V$$

consists of singular points of X. Since $\dim S \geq (n-1) - 4 = n - 5$ this leads to a contradiction. Thus the claim. □

Proposition 2. *Let $X \subset V$ be a closed cone, not contained in a linear subspace of V and let $\mathbb{P}(X) \subset \mathbb{P}(V)$ be its image. Assume that X or, equivalently, $\mathbb{P}(X)$ is smooth in codimension 1 and that the divisor class group* $\operatorname{Cl} X$ *is trivial or, equivalently, that* $\operatorname{Cl} \mathbb{P}(X)$ *is generated by a hyperplane section. Then $H \cap X$ and $D \cap \mathbb{P}(X)$ are irreducible for any linear hyperplane $H \subset V$ respectively. $D \subset \mathbb{P}(V)$. In particular, X and $\mathbb{P}(X)$ have the strong separation property* (SSP).

Proof. There is a canonical exact sequence

$$0 \to \mathbb{Z} \to \operatorname{Cl} \mathbb{P}(X) \to \operatorname{Cl} X \to 0$$

where the first map sends 1 to the class of $D \cdot \mathbb{P}(X)$, called a *hyperplane section* (see [Har77, Exercise II.6.3(b)]. This implies that the assumptions for X and $\mathbb{P}(X)$ are equivalent.

Let $\eta\colon \tilde{X} \to X$ be the normalization. Then $\operatorname{Cl} \tilde{X}$ is trivial by assumption and Lemma 1 and so $\tilde{X}$ is factorial (cf. [Har77, Proposition II.6.2]). Since the coordinate ring $\mathcal{O}(\tilde{X})$ is graded, we see that for every linear function ℓ on V the restriction $\ell|_X$ is an irreducible element of $\mathcal{O}(\tilde{X}) \supseteq \mathcal{O}(X)$. Thus $H \cap X$ is irreducible for every linear hyperplane of V and the same holds for $D \cap \mathbb{P}(X)$. The rest follows from Lemma 2. □

Remark 2. If we assume, in addition, that X is *normal*, then X is factorial and so the schematic intersection $X \cap H$ is irreducible *and reduced.* In general, $X \cap H$ is reduced in all points x where X is normal.

We end this section with some general results about the separation properties and a final example which will be used later in the paper.

Proposition 3. *Let $p\colon V \to W$ be a surjective linear map between K-vector spaces.*

1. *If $X \subset V$ has the separation property* (SP) *or the weak separation property* (WSP) *then so does $p(X) \subset W$.*
2. *Let $X \subset V$ be a closed cone such that $X \cap \ker p = \{0\}$. If X has the strong separation property* (SSP) *then so does $p(X) \subset W$.*

Proof. The first claim is easy and left to the reader. For the second, we first remark that $p(X)$ is a closed cone in W and that $p|_X\colon X \to p(X)$ is a finite morphism. If $L \subset W$ is any linear subspace and $\tilde{L} := p^{-1}(L)$ its inverse image in V then $p(X \cap \tilde{L}) = p(X) \cap L$. Hence, $\dim X \cap \tilde{L} = \dim p(X) \cap L$ and the claim follows. □

Remark 3. Let V, W be two finite dimensional vector spaces of positive dimension and $X \subset V$ a subset. Set $X \otimes W := \{x \otimes w \mid x \in X, w \in W\}$. *Then $X \subset V$ has the separation property* (SP) *if and only if $X \otimes W \subset V \otimes W$ does.*

As an application, we see that if $X \subset \bigwedge^k V$ has the separation property (SP) then $X \wedge V := \{x \wedge v \mid x \in X, v \in V\} \subset \bigwedge^{k+1} V$ satisfies (SP).
(Use the surjective homomorphism $\bigwedge^k V \otimes V \to \bigwedge^{k+1} V$ and Proposition 3(1) above.)

Example 4. Let K be an infinite field of characteristic $\neq 2$, $V = K^n$ $(n \geq 4)$ and q an integer > 1. For $a = (a_1, a_2, \dots, a_n) \in V$ define $\varphi(a) := (a_1^q, a_2^q, \dots, a_n^q)$. Put

$$X := \{\varphi(a) \wedge a \wedge b \mid a, b \in V\} \subset \bigwedge\nolimits^3 V.$$

Then X has the separation property (SP)*.*

Proof. For $\lambda, \mu \in (\bigwedge^3 V)^*$ we define $\lambda_{ijk} := \lambda(e_1 \wedge e_j \wedge e_k)$ and similarly for μ where $(e_1, e_2, \dots, e_n)$ is the standard basis of $V = K^n$. Then

$$f(a,b) := \lambda(\varphi(a) \wedge a \wedge b) = \sum_{i<j,k} a_i a_j (a_i^{q-1} - a_j^{q-1}) b_k \lambda_{ijk},$$
$$g(a,b) := \mu(\varphi(a) \wedge a \wedge b) = \sum_{i<j,k} a_i a_j (a_i^{q-1} - a_j^{q-1}) b_k \mu_{ijk}.$$

Now assume that $\lambda(x) = 0, x \in X$ implies $\mu(x) = 0$, i.e. $f(a,b) = 0$ implies $g(a,b) = 0$. Since $f(a_i e_i + a_j e_j, e_k) = a_i a_j (a_i^{q-1} - a_j^{q-1}) \lambda_{ijk}$ $(i < j)$ and similarly for g we first see that $\lambda_{ijk} = 0$ implies that $\mu_{ijk} = 0$. Now consider the linear functions $f(a, \cdot)$ and $g(a, \cdot)$. By assumption, the kernel of $g(a, \cdot)$ contains the kernel of $f(a, \cdot)$ and so $g(a, \cdot)$ is a scalar multiple of $f(a, \cdot)$, for all $a \in V$. Thus we obtain the following equations:

$$h_{ij}(a) := f(a, e_i) g(a, e_j) - f(a, e_j) g(a, e_i) = 0.$$

Expanding $h_{ij}(a)$ into monomials in $a_1, a_2, \dots, a_n$ we find that the coefficient of $a_r^{q+1} a_s^{q+1}$ equals $2(\lambda_{rsj}\mu_{rsi} - \lambda_{rsi}\mu_{rsj})$. Together with our first remark above this shows that there is a $c_{rs} \in K$ such that $\mu_{rsi} = c_{rs}\lambda_{rsi}$ for all i. Note that c_{rs} is uniquely defined if there is an i such that $\lambda_{rsi} \neq 0$. From this we conclude that if $r < s < i$ and $\lambda_{rsi} \neq 0$ then $c_{rs} = c_{ri} = c_{si}$. It also follows that if λ_{ijk} and λ_{rst} are both non-zero and have two indices in common then all c_{pm} for $p, m \in \{i, j, k, r, s, t\}$ are equal.

Next we consider the case where λ_{ijk} and λ_{rst} are both non-zero and have one index in common. By symmetry we can assume that $\lambda_{ijk} = \lambda_{123}$ and $\lambda_{rst} = \lambda_{145}$. Calculating the coefficient of $a_1^q a_3 a_4^q a_5$ in $h_{12}(a_1e_1 + a_2e_2 + \cdots + a_5e_5)$ we find

$$\lambda_{145}\mu_{123} + \lambda_{135}\mu_{124} - \lambda_{123}\mu_{145} - \lambda_{124}\mu_{135}.$$

Similarly, the coefficient of $a_1^q a_2 a_4 a_5^q$ in $h_{13}(a_1e_1 + a_2e_2 + \cdots + a_5e_5)$ is

$$\lambda_{145}\mu_{123} - \lambda_{135}\mu_{124} - \lambda_{123}\mu_{145} + \lambda_{124}\mu_{135}.$$

Hence, the sum of the two is $2(\lambda_{123}\mu_{145} - \lambda_{145}\mu_{123})$. Thus, there is a uniquely determined c such that $\mu_{123} = c\lambda_{123}$ and $\mu_{145} = c\lambda_{145}$. Since λ_{123} and λ_{145} are both non-zero, we obtain $c_{23} = c_{45}$. It follows again that all c_{pm} for $p, m \in \{i, j, k, r, s, t\}$ are equal in this case.

We are left with the case where λ_{ijk} and λ_{rst} are both non-zero and have no index in common. We can also assume that all other λ_{uvw} with $\{u, v, w\} \subset \{i, j, k, r, s, t\}$ are zero, since otherwise one of the previous cases applies. Setting $b := b_ie_i + b_je_j + b_ke_k + b_re_r + b_se_s + b_te_t$ we find

$$\begin{aligned} f(a, b) &= a_ia_j(a_i^q - a_j^q)b_k\lambda_{ijk} + a_ra_s(a_r^q - a_s^q)b_t\lambda_{rst}, \\ g(a, b) &= a_ia_j(a_i^q - a_j^q)b_k\mu_{ijk} + a_ra_s(a_r^q - a_s^q)b_t\mu_{rst}. \end{aligned}$$

Hence, there is a uniquely determined c such that $\mu_{ijk} = c\lambda_{ijk}$ and $\mu_{rst} = c\lambda_{rst}$ and the claim follows. ☐

Question 1. (Notation of Example 4) *Does the subset* $X := \{\varphi(a) \wedge a \mid a \in V\} \subset \bigwedge^2 V$ *have the separation property* (SP)*?*

§4. SEPARATION PROPERTIES FOR ORBITS

In this section we study the separation properties for orbits in representation spaces of algebraic groups. Let us first recall some basic facts. For a general reference we refer to [Jan87].

We assume that our base field K is algebraically closed of arbitrary characteristic $p \geq 0$. Let G be a semisimple, connected and simply connected algebraic group over K. We fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$ and denote by $X := X(T)$ the character group of T which can be identified with the character $X(B)$ group of B. Let $R \subset X(T)$ be the root system of (G, T), $R^+ \subset R$ the positive roots with respect to B and $\Delta = \{\alpha_1, \dots, \alpha_r\} \subset R^+$ the set of simple roots. We denote by $X^+ = X(T)^+$ the dominant integral weights, i.e. $X^+ = \{\lambda \in X(T) \mid (\lambda, \alpha^\vee) \geq 0 \text{ for all } \alpha \in \Delta\}$ where $\{\alpha^\vee \mid \alpha \in R\}$ denotes the set of coroots. The fundamental weights $\{\omega_1, \omega_2, \dots, \omega_r\}$ are defined by $(\omega_i, \alpha_j^\vee) = \delta_{ij}$, and we get $X^+ = \sum_{i=1}^r \mathbb{N}\omega_i$.

For $\lambda \in X(T)$ we denote by $\mathcal{L}(\lambda)$ the corresponding line bundle over G/B. Its total space is $G \times^B K_\lambda$ where K_λ denotes the one-dimensional representation of B with character λ. If $\lambda \in X^+$ then $\mathcal{L}(-\lambda)$ is very ample on G/B. The dual space $H^0(G/B, \mathcal{L}(-\lambda))^*$ is the Weyl module $V(\lambda)$.

If M is an arbitrary (finite dimensional) B-module, then the induced G-module $\operatorname{Ind}_B^G(M)$ is given by $(\mathcal{O}(G) \otimes M)^B = \operatorname{Mor}_B(G, M)$ where B acts on the right of G. Let $\mathcal{V}(M)$ denote the vector bundle over G/B associated to the B-module M, with underlying space $G \times^B M$. Then

$$\operatorname{Ind}_B^G(M) = \operatorname{Mor}_B(G, M) \simeq H^0(G/B, \mathcal{V}(M)).$$

Hence $H^0(G/B, \mathcal{L}(\lambda)) \simeq \operatorname{Ind}_B^G(K_\lambda)$ and so $V(\lambda)$ is the dual of the induced module $\operatorname{Ind}_B^G(K_{-\lambda})$.

If $p\colon G \times^B M \to G/B$ is the projection then $p^*(\mathcal{O}_{G \times^B M}) \simeq \bigoplus_{n \geq 0} \mathcal{V}(S^n M^*)$ and so

$$\mathcal{O}(G \times^B M) \simeq \bigoplus_{n \geq 0} H^0(G/B, \mathcal{V}(S^n M^*)) \simeq \bigoplus_{n \geq 0} \operatorname{Ind}_B^G(S^n M^*).$$

In particular, $\mathcal{O}(G \times^B K_\lambda) = \bigoplus_{n \geq 0} V(n\lambda)^*$.

If W is an arbitrary G-module and $\mu \in X$ we denote by $W_\mu \subset W$ the weight space of weight μ. The character of W is given by the formal sum $\sum_{\mu \in X} (\dim W_\mu) \cdot e^\mu$. It is well-known that the character of the Weyl modules $V(\lambda)$ ($\lambda \in X^+$) is given by the Weyl character formula.

If $\operatorname{char} K = 0$, then $V(\lambda)$ is irreducible of highest weight λ. In general, a G-module W is called a *highest weight module* if W contains a B-stable line Kw such that $W = \operatorname{Span}_K(Gw)$. If λ is the weight of $K\lambda$ then $\lambda \in X^+$ and W is called a *module of highest weight* λ. It is not difficult to see that λ is indeed the highest weight of W and that W is a quotient of the Weyl module $V(\lambda)$. Moreover, the Weyl module $V(\lambda)$ (more generally, every highest weight module) is indecomposable and has a unique simple quotient $L(\lambda)$.

If W is a module of highest weight λ then the orbit of a highest weight vector $w_\lambda \in W_\lambda$ ($w_\lambda \neq 0$) is called *minimal orbit* of W and will be denoted by $O_{\min,W}$ or shortly by $O_{\min}$. Let P_λ denote the parabolic subgroup generated by B and all root subgroups $U_{-\alpha}$ such that $(\lambda, \alpha^\vee) = 0$. Thus $P_\lambda = P_I$ where $I = \{\alpha \in \Delta \mid (\lambda, \alpha^\vee) = 0\}$. The subgroup P_λ stabilizes W_λ and we obtain a canonical morphism $\varphi\colon G \times^{P_\lambda} W_\lambda \to W$ which is proper and whose image is GW_λ.

Proposition 4. *Let W be a module of highest weight $\lambda \neq 0$ and $O_{\min} \subset W$ the minimal orbit.*

(1) $\overline{O_{\min}} = GW_\lambda = O_{\min} \cup \{0\}$.

(2) *If* $w \in W$ *is not contained in a proper submodule then* $O_{\min} \subset \overline{K \cdot Gw}$.
(3) *If* W *is irreducible then* $O_{\min}$ *corresponds to the only closed orbit in* $\mathbb{P}(W)$.
(4) *If* W *equals the Weyl module* $V(\lambda)$ *then* $\overline{O_{\min}}$ *is normal with rational singularities.*
(5) *The stabilizer of the line* $V(\lambda)_\lambda$ *is the reduced parabolic subgroup* P_λ. *In particular,* $\varphi\colon G \times^{P_\lambda} V(\lambda)_\lambda \to \overline{O_{\min}} \subset V(\lambda)$ *is proper and birational.*
(6) *If* $p\colon V(\lambda) \to W$ *is a quotient then the induced map* $\overline{O_{\min,V(\lambda)}} \to \overline{O_{\min,W}}$ *is bijective.*
(7) *Assume that the stabilizer* P *of the line* W_λ *is reduced and that* $\overline{O_{\min}}$ *is normal. Then* $W \simeq V(\lambda)$ *as a* G*-module.*

Outline of proof. The first statement (1) is well-known. For (2) we first remark that the orbit Gw contains an element w' whose component w'_λ in the weight space W_λ is non-zero. Let $\rho\colon K^* \to G$ be a 1-parameter subgroup such that $m := (\lambda, \rho) > (\mu, \rho)$ for all weights $\mu \neq \lambda$ of W. Then $\lim_{t\to 0} t^m \cdot \rho(t^{-1})w' = w'_\lambda$ and so $O_{\min} = Gw'_\lambda \subset \overline{K \cdot Gw}$.

Statement (3) is an immediate consequence of (2).

Statement (4) is well-known in characteristic 0 (see [ViP72]). In positive characteristic it follows from the work of RAMANAN-RAMANATHAN [RaR85, Theorem 3] and RAMANATHAN [Ra85] showing that the linear system on G/P where $P \subset G$ is a parabolic subgroup, given by any ample line bundle embeds G/P as a projectively normal and Cohen-Macaulay variety (see Appendix, Theorem A).

For (5) we remark that the ample line bundle $\mathcal{L}(\lambda)$ on G/P_λ embeds G/P_λ into $\mathbb{P}(V(\lambda))$ as the orbit of the highest weight line $V(\lambda)_\lambda \in \mathbb{P}(V(\lambda))$.

Statement (6) follows from the fact that the reduced stabilizer of the line $W_\lambda \subset W$ is the parabolic subgroup P_λ.

For the last statement (7) we remark that, by assumption, the canonical morphism $\varphi\colon G \times^P W_\lambda \to \overline{O_{\min}} \subset W$ is birational (and proper). Since $\overline{O_{\min}}$ is normal the induced map $\varphi^*\colon \mathcal{O}(\overline{O_{\min}}) \xrightarrow{\sim} \mathcal{O}(G \times^P W_\lambda)$ is an isomorphism. Now the claim follows from the isomorphism $\mathcal{O}(G \times^P W_\lambda) \simeq \mathcal{O}(G \times^B K_\lambda) \simeq \bigoplus_{n\geq 0} V(n\lambda)^*$ mentioned above. ☐

Remark 4. It follows from Proposition 4(5) that the stabilizer of a highest weight vector $v \in V(\lambda)_\lambda$ is reduced if $\lambda \notin pX^+$ where $p := \operatorname{char} K$. (In fact, $G_v = \ker(\lambda\colon P_\lambda \to K^*)$, and the differential of $\lambda|_T$ is surjective under the given assumptions.)

In general, the stabilizer of the highest weight line W_λ in a highest weight module W is not necessarily reduced. A description of these stabilizers for the simple modules $L(\lambda)$ is given in [Lau96]. Under mild assumptions about the characteristic it follows from this description that the stabilizer of $L(\lambda)_\lambda$ is reduced if and only if $(\lambda, \alpha^\vee) < p$ for all simple roots $\alpha \in \Delta$. As a consequence, we obtain the following non-normality result:

Assume that $(\lambda, \alpha_i^\vee) < p$ *for all simple roots* α_i. *If* $L(\lambda)$ *is a proper quotient of the Weyl module then* $\overline{O_{\min}} \subset L(\lambda)$ *is not normal.*

Remark 5. For later applications we describe another way to construct the Weyl modules. Let $\lambda = \sum_{i=1}^{\ell} n_i\omega_i \in X^+$ be a highest weight expressed in terms of fundamental weights, and let $v_i \in V(\omega_i)$ be highest weight vectors, for $i = 1, \dots, \ell$. Define $v := v_1 \otimes v_2 \otimes \cdots \otimes v_\ell \in V(\omega_1) \otimes V(\omega_2) \otimes \cdots \otimes V(\ell)$. Then $\operatorname{Span}_K(Gv) \simeq V(\lambda)$. This follows, by dualizing, from [Jan87, II. Proposition 14.20].

For a given $\lambda \in X^+$ we have $\lambda = \sum_i (\lambda, \alpha_i^\vee)\omega_i$. Set $J := \{j \mid (\lambda, \alpha_j^\vee) \neq 0\}$. Then the character group of P_λ considered as a subgroup of $X(T)$ has the following description: $X(P_\lambda) = \sum_{j\in J} \mathbb{Z}\omega_j$. It is well known that the Picard group of G/P_λ is canonically isomorphic to $X(P_\lambda)$: a character χ corresponds to the induced line bundle $G \times^{P_\lambda} K_\chi$ on G/P_λ ([FoI73], cf. [KKV89]). It is easy to see that a hyperplane section $D \cap G/P_\lambda$ in the canonical embedding $G/P_\lambda \subset \mathbb{P}(V(\lambda))$ corresponds to the character $\lambda \in X(P_\lambda)$. Therefore, $D \cap G/P_\lambda$ is irreducible (and reduced) in case λ is a fundamental weight. Moreover, the divisor $D \cdot G/P_\lambda$ is reduced if all $(\lambda, \alpha_i^\vee)$ are ≤ 1. Using Lemma 2 and Proposition 3 this proves the following result.

Proposition 5. *Let W be a module of highest weight $\lambda = \sum_i m_i\omega_i$.*

1. *If λ is a fundamental weight then $\overline{O_{\min}} \subset W$ has the strong separation property* (SSP). *More precisely, for every hyperplane H of W the intersection $W \cap \overline{O_{\min}}$ is irreducible and reduced.*
2. *If $m_i \leq 1$ for all i then $O_{\min}$ has the separation property* (SP). *More precisely, for every hyperplane H of W the divisor $H \cdot \overline{O_{\min}}$ is reduced.*

We will see in the following Theorem 1 that for Weyl modules $V(\lambda)$ the given assumptions are also necessary for the separation properties to hold.

The proposition implies the strong separation property (SSP) for to the adjoint representations of the simple groups of type $\mathtt{B}_n$ $(n \geq 3)$, $\mathtt{D}_n$ $(n \geq 4)$, $\mathtt{E}_n$ $(n = 6, 7, 8)$, $\mathtt{F}_4$ and $\mathtt{G}_2$, and the separation property (SP) for the adjoint representation of SL_n $(n > 2)$. It does not apply to the adjoint representations of Sp_{2n} as shown by the following example.

Example 5. Consider the representation $V := S^d(K^n)$ of $G = \mathrm{SL}_n$ and assume that $\operatorname{char} K = 0$ or $> d$. Then V is irreducible and $O_{\min} = \{x^d \mid x \in K^n\}$. *We claim that for $d > 1$ the orbit $O_{\min}$ does not have the separation property* (SP). In particular, the minimal conjugacy class in $\mathfrak{sp}_{2n}$ does not have the separation property ($\operatorname{char} K \neq 2$). (Recall that $\mathfrak{sp}_{2m} \simeq S^2(K^{2m})|_{\mathrm{Sp}_{2m}}$.)

Proof. Using the canonical isomorphism of $V^* \simeq \mathcal{O}(K^n)_d$ a linear function $\alpha \in V^*$ can be regarded as a homogeneous polynomial f_α of degree d, and we have $\alpha(v^d) = f_\alpha(v)$. Now we choose two linearly independent linear functions ℓ_1 and ℓ_2 on K^n and define $\alpha := \ell_1^d$ and $\beta := \ell_1^{d-1}\ell_2$. Clearly, $\alpha(x) = 0$ for $x = v^d$ implies that $\beta(x) = 0$, but β and α are linearly independent. ☐

Now we come to our first main result.

Theorem 1. *Let G be a connected semisimple group and let $\lambda = \sum_i n_i\omega_i$ be a dominant integral weight expressed in terms of fundamental weights. As before, we denote by $O_{\min} \subset V(\lambda)$ the minimal orbit $\neq \{0\}$ in the corresponding Weyl module of highest weight λ.*

1. *$\overline{O_{\min}}$ satisfies* (SSP) $\iff$ *λ is a fundamental weight.*
2. *$O_{\min}$ satisfies* (SP) $\iff$ *$n_i \leq 1$ for all i.*
3. *$O_{\min}$ satisfies* (WSP) $\iff$ *$n_i \leq 2$ for all i.*

Proof. Consider the morphism

$$\varphi\colon V(\omega_1) \times V(\omega_2) \times \cdots \times V(\omega_\ell) \to V(\omega_1)^{\otimes n_1} \otimes V(\omega_2)^{\otimes n_2} \otimes \cdots \otimes V(\omega_\ell)^{\otimes n_\ell}$$

where $\varphi(v_1, v_2, \ldots, v_\ell) := v_1^{\otimes n_1} \otimes v_2^{\otimes n_2} \otimes \cdots \otimes v_\ell^{\otimes n_\ell}$. (We omit those ω_i where $n_i = 0$ and assume therefore that all $n_i \geq 1$.)

Let $u_i \in V(\omega_i)$ be highest weight vectors. Then $u := \varphi(u_1, u_2, \ldots, u_\ell)$ is a highest weight vector of the Weyl module $V(\lambda) = \operatorname{Span}_K(Gu) \subset V(\omega_1)^{\otimes n_1} \otimes V(\omega_2)^{\otimes n_2} \otimes \cdots \otimes V(\omega_\ell)^{\otimes n_\ell}$ (see Remark 5). Moreover, the torus ${K^*}^\ell$ acts on $V(\omega_1) \times V(\omega_2) \times \cdots \times V(\omega_\ell)$ in the obvious way. An easy calculation shows that φ^* induces a surjective homomorphism

$$(*) \qquad \mathcal{O}(V(\omega_1)^{\otimes n_1} \cdots \otimes V(\omega_\ell)^{\otimes n_\ell})_1 \twoheadrightarrow \mathcal{O}(V(\omega_1) \times \cdots \times V(\omega_\ell))_{(n_1, n_2, \ldots, n_\ell)}$$

where $\mathcal{O}(V(\omega_1) \times \cdots \times V(\omega_\ell))_{(n_1,n_2,\ldots,n_\ell)}$ denotes the homogeneous component of multidegree $(n_1, n_2, \ldots, n_\ell)$. Consider the orbit $O := G(u_1, u_2, \ldots, u_\ell)$ and its closure $Y := \overline{O} \subset V_{\omega_1} \times V_{\omega_2} \times \cdots \times V_{\omega_\ell}$. Both are stable under ${K^*}^\ell$. Therefore, the coordinate ring $\mathcal{O}(Y)$ is also r-multigraded. Moreover, Y is normal and factorial (see Appendix, Proposition B).

By definition, $\varphi(O) = O_{\min} = Gu \subset V(\lambda)$ and we get a surjective morphism $\psi := \varphi|_Y \colon Y \twoheadrightarrow \overline{O_{\min}}$. It follows from $(*)$ that the induced homomorphism

$$\psi^*\colon V(\lambda)^* = \mathcal{O}(\overline{O_{\min}})_1 \xrightarrow{\sim} \mathcal{O}(Y)_{(n_1,n_2,\ldots,n_\ell)}$$

is an isomorphism.

Let f, h be two linear functions on $V(\lambda)$ such that $f(x) = 0$ for $x \in O_{\min}$ implies that $h(x) = 0$, i.e., that the zero set $\mathcal{V}(h) \subset \overline{O_{\min}}$ contains $\mathcal{V}(f)$. Then the same holds for the functions $\tilde{f} := \varphi^*(f)$ and $\tilde{h} := \varphi^*(h)$ on Y which are multihomogeneous of degree $(n_1, n_2, \ldots, n_r)$:

$$\mathcal{V}_{\overline{Y}}(\tilde{f}) \subseteq \mathcal{V}_{\overline{Y}}(\tilde{h}).$$

Since $\mathcal{O}(Y)$ is factorial this means that every irreducible factor of $\tilde{f}$ occurs in $\tilde{h}$. Moreover, every such factor is multihomogeneous, too.

First assume that all the n_i are equal to one. Then $\tilde{f}$ and $\tilde{h}$ are squarefree and of the same multidegree $(1, 1, \ldots, 1)$. It follows that $\tilde{h}$ is a scalar multiple of $\tilde{f}$ and therefore h is a multiple of f. This shows that $O_{\min} \subset V_\lambda$ has the separation property in this case. Moreover, for $r \geq 2$ the orbit $O_{\min}$ cannot have the strong separation property. In fact, let $\alpha_i \in V(\omega_i)^*$ be non-zero linear functions, $i = 1, 2, \dots, \ell$, and let $\beta_1 \in V(\omega_1)^*$ be linearly independent from α_1. Then $\tilde{f} := \alpha_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_\ell$ and $\tilde{h} := \beta_1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_\ell$ are pullbacks from linearly independent functions $f, h \in V(\lambda)^*$. It follows that $F := \psi^{-1}(\mathcal{V}(f) \cap \mathcal{V}(h)) = \mathcal{V}(\tilde{f}, \tilde{h}) \supset \mathcal{V}(1 \otimes \alpha_2 \otimes \cdots \otimes \alpha_\ell)$ has codimension 1 in Y. Since $\operatorname{codim}_Y(Y \setminus Gv) \geq 2$, the subvariety F meets the orbit Gv in a hypersurface $F' := F \cap Gv$. Thus, $\mathcal{V}(f) \cap \mathcal{V}(h)$ contains $\psi(F')$ which is a hypersurface in $\psi(Gv) = O_{\min} \subset V(\lambda)$.

Now assume that not all n_i are equal to 1, e.g., assume that $n_1 > 1$, and let $\alpha_i \in V(\omega_i)^*$, $\beta_1 \in V(\omega_1)^*$ be as above. Then $\tilde{f} := \alpha_1^{n_1}\alpha_2^{n_2}\cdots\alpha_\ell^{n_\ell}$ and $\tilde{h} := \alpha_1^{n_1-1}\beta_1\alpha_2^{n_2}\cdots\alpha_\ell^{n_\ell}$ both belong to $\mathcal{O}(Y)_{(n_1,n_2,\dots,n_\ell)}$ and are therefore pullbacks of linear functions $f, h \in V(\lambda)^*$ which are linearly independent. But $\tilde{h}^2$ is a multiple of $\tilde{f}$ and so $\mathcal{V}(f) = \psi(\mathcal{V}_Y(\tilde{f})) \supset \psi(\mathcal{V}_Y(\tilde{h})) = \mathcal{V}(h)$.

Finally, assume that $n_i \leq 2$ for all i and let $\mathcal{V}(f) = \mathcal{V}(h)$. Then the pull backs $\tilde{f}$ and $\tilde{h}$ have the same irreducible factors:

$$\tilde{f} = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k} \quad \text{and} \quad \tilde{h} = p_1^{s_1}p_2^{s_2}\cdots p_k^{s_k}$$

where $1 \leq r_i, s_i \leq 2$ for all i. If $r_i = 2$ then p_i has multidegree $d = (d_1, d_2, \dots, d_\ell)$ where $d_k \in \{0, 1\}$. Moreover, d is "disjoint" from the multidegree of any other p_j, i.e., if the degree of p_i in the kth component is 1, then the degree of all other p_j's in the kth component is zero. This obviously implies that $s_i = 2$ since $\tilde{f}$ and $\tilde{h}$ have the same multidegree. Thus $\tilde{h}$ is a scalar multiple of $\tilde{f}$ and so h is a scalar multiple of f.

Conversely, if $r_1 > 2$, we put $\tilde{f} := \alpha_1^{r_1-1}\beta_1 \otimes \alpha_2^{r_2} \otimes \cdots$ and $\tilde{h} := \alpha_1\beta_1^{r_1-1} \otimes \alpha_2^{r_2} \otimes \cdots$. Then $\tilde{f}$ and $\tilde{h}$ are pullbacks from linearly independent linear functions f and $h \in V(\lambda)^*$ and, by construction, $\mathcal{V}(f) = \mathcal{V}(h)$. □

Corollary 1. *Consider the adjoint representation of a simple group G on its Lie algebra $\mathfrak{g}$ and denote by $C_{\min} \subset \mathfrak{g}$ the minimal nilpotent conjugacy class. Then $C_{\min}$ has the weak separation property* (WSP)*. Moreover, the separation property* (SP) *holds for all G except those of type* `A`$_1$ *and* `C`$_n$ ($n \geq 2$) *and the strong separation property* (SSP) *for all G except those of type* `A`$_n$ *and* `C`$_n$.

Let $\operatorname{char} K = p > 0$. If W is a G-module we denote by $W^{[r]}$ the rth *Frobenius twist* of W, i.e. the module obtained by composing the representation $G \to \mathrm{GL}(W)$ with the rth Frobenius homomorphism $F^r \colon G \to G$. (We can assume that G is defined over $\mathbb{F}_p$.) If W is a module of highest weight λ then $W^{[r]}$ is a module of highest weight $p^r\lambda$. Clearly, the different separation properties hold for the minimal orbit in W if and only if they hold for the minimal orbit in $W^{[r]}$.

The next result is an immediate consequence of the theorem above, combined with Propositions 3 and 4(6).

Corollary 2. *Let W be a module of highest weight λ and let $O_{\min} \subset W$ be the minimal orbit.*

1. *If $\lambda = p^r\omega_i$ with some $r \geq 0$ then $\overline{O_{\min}}$ has the strong separation property* (SSP)*.*
2. *If there is an $r \geq 0$ such that $(\lambda, \alpha^\vee) = 0$ or p^r for all simple roots α then $O_{\min}$ has the separation property* (SP)*.*
3. *If there is an $r \geq 0$ such that $(\lambda, \alpha^\vee) = 0$, p^r or $2p^r$ for all simple roots α then $O_{\min}$ has the weak separation property* (WSP)*.*

Theorem 1 gives necessary and sufficient conditions to decide if the minimal orbit in Weyl module has one of the separation properties. Thus we have complete solution in characteristic zero. In positive characteristic the situation is more complicated, due to the Frobenius twist. The following example shows that the list of simple G-modules with the separation property (SP) given in Corollary 2 is not complete. In fact, we conjecture the following. Let $\operatorname{char} K = p > 0$ and set $X_p^+ := \{\mu \in X^+ \mid (\mu, \alpha^\vee) < p \text{ for all simple roots } \alpha\}$.

Conjecture. *Let $\lambda_1, \lambda_2, \ldots, \lambda_r \in X_p^+$ and assume that $\lambda := \lambda_1 + \lambda_2 + \cdots + \lambda_r$ satisfies the condition $(\lambda, \alpha^\vee) \leq 1$ for all simple roots α. If $0 \leq m_1 < m_2 < \cdots < m_r$ is an increasing sequence of integers then the simple module $L(p^{m_1}\lambda_1 + p^{m_2}\lambda_2 + \cdots + p^{m_r}\lambda_r)$ has the separation property* (SP)*.*

Example 6. Let $V := K^n$ and $G = \mathrm{SL}(V)$. Assume that the characteristic of K is different from 2. Fix a number $1 < m < n$ and consider the G-modules $M_r := V^{[r]} \otimes \bigwedge^m V$ for $r \geq 0$. By STEINBERG's Theorem (see [Jan87, II.3.17]) this module is irreducible for $r > 0$.

CLAIM. *The minimal orbit in M_r has the separation property* (SP) *for $r > 0$.*

Proof. The closure of the minimal orbit $O_r \subset M_r$ has the following description:

$$O_r = \{F^r(v_1) \otimes (v_1 \wedge v_2 \wedge v_3 \cdots \wedge v_m) \mid v_1, \ldots, v_m \in V\}.$$

Now we use the following result: *If, for some r, the linear forms $\lambda, \mu \in M^*$ have the property that $\lambda(x) = 0$ implies $\mu(x) = 0$ for any $x \in O_r$, then λ, μ have the same property for O_0.*

(Write λ, μ in terms of the standard basis of M. If $u, v_1, ..., v_{m-1}$ are in V and if $\xi \in M^*$ then we define

$$\xi_{u,v_1,\ldots,v_{m-1}}(w) := \xi(u \otimes (v_1 \wedge v_2 \wedge v_3 \cdots \wedge v_{m-1} \wedge w)).$$

Our assumption is that for all choices of $v_1, ..., v_{m-1}$ the two linear functions $\lambda_{F^r(v_1),v_1,\ldots,v_{r-1}}$ and $\mu_{F^r(v_1),v_1,\ldots,v_{r-1}}$ are linearly dependent. If we write out the corresponding 2×2 determinants (that is the coefficients of $\lambda_{F^r(v_1),v_1,\ldots,v_{r-1}} \wedge \mu_{F^r(v_1),v_1,\ldots,v_{r-1}}$) and expand in monomials in the coordinates of the v_i then the coefficients of the monomials for $r = 0$ are sums of those for our fixed $r > 0$.)

We know from Theorem 1 that the highest weight module $L := \operatorname{Span}_K(O_0) \subset M_0$ has the separation property (SP). Thus we get $\mu|_L = c\lambda|_L$ with $c \in K$. Replacing μ by $\mu - c\lambda$, we may assume that $\mu|_L = 0$. We wish to conclude that $\mu = 0$. We next observe that we have the exact sequence of G-modules

$$0 \to L \to M_0 \xrightarrow{p} \bigwedge^{m+1} V \to 0$$

with the first arrow the canonical injection and the last the canonical projection. By construction, $\mu = \pi^* \nu$ for some $\nu \in \bigwedge^{m+1} V$. We now note that if v_1 is given and if $F^r(v_1) \notin \ker \lambda_{F^r(v_1), v_1, \dots, v_{m-1}}$ then $\nu(F^r(v_1) \wedge v_1 \wedge v_2 \wedge v_3 \cdots \wedge v_{m-1} \wedge V) = 0$. It follows that there is a Zariski open subset $\Omega \subset V^n$ such that $\nu(F^r(v_1) \wedge v_1 \wedge v_2 \wedge v_3 \cdots \wedge v_{m-1} \wedge v_m) = 0$ for $(v_1, \dots, v_m) \in \Omega$. This implies that $\nu = 0$ and the result is proved in this case.

We are therefore left with the case when $F^r(v_1) \in \ker \lambda_{F^r(v_1), v_1, \dots, v_{m-1}}$ for all choices of $v_1, \dots, v_{m-1}$. This implies that λ vanishes on the span of the elements

$$F^r(v_1) \otimes v_1 \wedge v_2 \cdots \wedge v_{m-1} \wedge F^r(v_1).$$

This span is L. Thus $\lambda = \pi^* \xi$ for some $\xi \in \bigwedge^{m+1} V$. We conclude that if $v_1, \dots, v_m \in V$ then

$$\xi(F^r(v_1) \wedge v_1 \wedge v_2 \wedge \cdots \wedge v_m) = 0$$

implies that

$$\nu(F^r(v_1) \wedge v_1 \wedge v_2 \wedge \cdots \wedge v_m) = 0.$$

Now the claim follows from Example 4. ☐

Question 2. Let G be a reductive group and V a highest weight module of G. Recall that the *nullcone* of V is the G-stable closed subset $\mathcal{N}$ of V defined by the vanishing of all homogeneous invariants.
Does the nullcone $\mathcal{N} \subset V$ *have the separation property? Or the weak separation property, or even the strong separation property?*

§5. SEPARATION PROPERTY FOR REPRESENTATIONS

We now come back to the question whether a general orbit in a representation space has the separation property.

Definition 5. A representation $\rho\colon G \to \mathrm{GL}(V)$ of an algebraic group G has the *separation property* if every orbit in V different from $\{0\}$ has the separation property.

A representation V with the separation property has to be irreducible, since otherwise there are orbits $\neq \{0\}$ contained in linear hyperplanes. We now show that it suffices to check the minimal orbit $O_{\min}$.

Theorem 2. *Let V be an irreducible representation of a connected semisimple group G. Assume that the separation property holds for the minimal orbit $O_{\min}$ in V. Then the separation property holds for any G-orbit O. In particular, every hyperplane meets the orbit O.*

Proof. We can assume $\dim W \geq 2$ since there is nothing to prove for $\dim W = 1$. Let $O \subset V$ be a G-orbit, not contained in a proper submodule. Then $\overline{K^*O} \supset O_{\min}$ and so $\overline{K^*O}$ has the separation property. If O fails to have the separation property, we can find two linear subspaces $H \neq H'$ of codimension 1 such that $H \cap O \subset H'$. Then $H \cap \lambda X = \lambda(H \cap O) \subset H'$ for every $\lambda \neq 0$ and so $H \cap K^*O \subset H'$. It follows from Lemma 3(b) below that $H \cap \overline{K^*O} = \overline{H \cap K^*O} \subset H'$. But this contradicts the assumption since $\overline{K^*O}$ contains the minimal orbit $O_{\min}$ which has the separation property, by assumption. ☐

Lemma 3. *Let V be a representation of a connected algebraic group G and which does not admit 1-dimensional subquotients. Let $H \subset V$ be a linear subspace of codimension 1.*

(a) *H meets every non-empty G-stable subset X of V.*

Assume in addition that V is irreducible. Then

(b) *$H \cap X \neq X$ for any non-empty G-stable subset $X \neq \{0\}$.*
(c) *If $X \subset V$ is a constructible G-stable subset then $\overline{H \cap X} = H \cap \overline{X}$.*

Proof. (1) We first claim that if a representation W of a connected group G contains a 1-dimensional orbit then it also contains a 1-dimensional subrepresentation. In fact, if $O \subset W$ is such an orbit and $\overline{O}$ not a line, then K^*O is G-stable and of dimension 2 and so $Y := \overline{K^*O} \setminus K^*O$ is a G-stable 1-dimensional cone in W. Thus Y is a finite union of one-dimensional subrepresentations of W.

(2) Now let $O \subset V$ be an orbit which does not meet a linear hyperplane H. Then $K^*O \cap H = \emptyset$ and so we can replace the group G by the larger group $\tilde{G} := K^* \times G$ where K^* acts by scalar multiplication. Then $H \cap \overline{O}$ is of codimension 1 in $\overline{O}$ and therefore of dimension ≥ 1, by (1) and the assumption. Since $H \cap \overline{O} \subset \overline{O} \setminus O$ it follows that $H \cap \overline{O}$ is a union of irreducible components of $\overline{O} \setminus O$. Thus $H \cap \overline{O}$ is G-stable and linearly generates a G-stable subspace $W \subset H$ of dimension > 0. Putting $V_1 := V/W$, $H_1 := H/W$ and $O_1 :=$ image of O in V_1, then we find that the linear hyperplane H_1 does not meet O_1 and we can repeat the argument above. Since $\dim V_1 < \dim V$ this eventually leads to a contradiction. Hence claim (a). Statement (b) is clear because V is irreducible.

(3) For (c) we can assume that $\overline{X}$ is irreducible and that X is open and dense in $\overline{X}$. In fact, let $\overline{X} = \bigcup_i Y_i$ be the decomposition into irreducible components. Then every Y_i is irreducible and G-stable and we can find subsets $X_i \subset X \cap Y_i$ which are open and G-stable in $\overline{X_i} = Y_i$. If (b) holds for the X_i's then it also holds for X:

$$H \cap \overline{X} = \bigcup_i (H \cap \overline{X_i}) = \bigcup_i \overline{H \cap X_i} = \overline{\bigcup\nolimits_i (H \cap X_i)} = \overline{H \cap \bigcup_i X_i} \subseteq \overline{H \cap X}$$

If $\overline{X}$ is irreducible and X open in $\overline{X}$, then $Y := \overline{X} \setminus X$ is G-stable and closed in $\overline{X}$ and $\dim Y \leq \dim X - 1$. (We can assume that $\overline{X} \neq X$ and so $\dim X \geq 2$.) It follows that $\dim(H \cap Y) \leq \dim Y - 1 \leq \dim X - 2$ by (1). But $H \cap \overline{X} = \overline{H \cap X} \cup (H \cap Y)$ is a union of irreducible closed subvarieties of dimension $\dim X - 1$ and so $H \cap Y \subset \overline{H \cap X}$. The claim follows. ☐

Remark 6. For the following application to Lie algebras we remark that Lemma 3 (and therefore Proposition 5) holds also for G-modules V containing a trivial 1-dimensional submodule V_0 such that V/V_0 is irreducible of dimension at least 2 if we assume in addition that the G-stable set X has positive dimension.
(The only modification of the proof of the lemma is in the last part (3) where we claim that $\dim(H \cap Y) < \dim Y$. If $\dim(H \cap Y) = \dim Y$ (≥ 1) then $H \cap Y$ is G-stable and generates a proper submodule. Hence $\overline{H \cap Y} = V_0$ and so $\dim Y = 1$ and $\dim \overline{X} = 2$. But this is only possible for $\overline{X} = \overline{O_{\min}}$ and then $Y = \overline{X} \setminus X \subset \{0\}$, contradicting the assumption.) ☐

Concerning Lie algebras of simple groups we now get the following result.

Corollary 3. *Consider the adjoint representation of a simple group G on its Lie algebra $\mathfrak{g}$. Assume that G is not of type $\mathtt{A}_1$ or $\mathtt{C}_n$ $(n \geq 2)$, that* $\operatorname{char} K \neq 2$ *in case G is of type $\mathtt{B}_n$, $\mathtt{D}_n$ or $\mathtt{F}_4$ and $\neq 3$ in case G is of type $\mathtt{G}_2$. Then $\mathfrak{g}$ has the separation property* (SP) *for all conjugacy classes of positive dimension.*

Proof. The minimal conjugacy class $C_{\min}$ in the Lie algebras of the simple groups not of type $\mathtt{A}_1$ or $\mathtt{C}_n$ $(n \geq 2)$ has the separation property (SP) according to Corollary 1 of §4. It follows from [His84] that under the given assumptions about the characteristic the center $\mathfrak{z}(\mathfrak{g})$ has dimension ≤ 1 and the Lie algebras $g/\mathfrak{z}(\mathfrak{g})$ is a simple G-module. Now the claim follows from Remark 6. ☐

We finish this section by showing that a generic orbit in an irreducible representation always has the separation property (SP), at least in characteristic zero.

Theorem 3. *Assume* $\operatorname{char} K = 0$. *Let $O \subset V$ be a generic orbit of an irreducible representation of a semisimple group G. Then O has the separation property* (SP). *More precisely, the intersection $H \cap O$ with every hyperplane is reduced.*

Proof. It suffices to prove that $H \cap O$ reduced (Lemma 2).

(a) Fix an arbitrary hyperplane $H \subset V$, $H = \ker \lambda$, and define

$$Z := \{(X, v) \in \mathfrak{g} \times V \mid \lambda(Xv) = 0\}.$$

Then Z is an irreducible hypersurface of $\mathfrak{g} \times V$ of dimension $\dim \mathfrak{g} + n - 1$ where $n := \dim V$. (In fact, Z is defined by a bilinear function. If Z where reducible then Z would be the union of two hyperplanes, $Z = \mathfrak{g}' \times V \cup \mathfrak{g} \times V'$ where $\mathfrak{g}' \subset \mathfrak{g}$ and $V' \subset V$ are linear subspaces of codimension 1. This is clearly not possible.)

(b) Let K^* act on $\mathfrak{g} \times V$ by scalar multiplication on $\mathfrak{g}$. Then Z is stable under this action and the morphism $p \colon Z \to V$ is the quotient. Consider the composition quotient is $q\colon Z \xrightarrow{p} V \xrightarrow{\pi} V/\!\!/G$ where π is the usual quotient with respect to G. Since K^* and G are both connected the generic fiber of q is irreducible. This implies that $p^{-1}(O)$ is irreducible for a generic orbit $O \subset V$. (Recall that the generic fiber of $\pi\colon V \to V/\!\!/G$ is an orbit since G is semisimple.)

(c) Let O be a generic orbit so that $p^{-1}(O)$ is irreducible by (b). Then the fiber over a generic point of O is a linear subspace of $\mathfrak{g}$ of codimension 1. Define

$$O_1 := \{v \in O \mid \dim p^{-1}(v) > \dim \mathfrak{g} - 1\} = \{v \in O \mid p^{-1}(v) = \mathfrak{g}\}.$$

Then $\operatorname{codim}_O O_1 \geq 2$. Otherwise $\dim p^{-1}(O_1) = \dim O - 1 + \dim \mathfrak{g} = \dim p^{-1}(O)$ which is impossible since $p^{-1}(O)$ is irreducible.

(d) By construction, $p^{-1}(v) = \mathfrak{g}$ means that $T_vO = \mathfrak{g}v \subset H$. We also know that $O \not\subset H$ and that $H \cap V \neq \emptyset$ (Lemma 3(a)). Thus

$$O_2 := \{v \in O \cap V \mid H \cap V \text{ not smooth in } v\} \subset O_1$$

and so O_2 has codimension at least 1 in $H \cap O$. Since $H \cap O$ is Cohen-Macaulay it follows that $H \cap O$ is reduced. □

§6. SEPARATION PROPERTY FOR BINARY FORMS

This case has been studied by KARIN BAUR in her thesis [Bau01]. We assume that the base field K is of characteristic zero.

Theorem 4. *Let V_n denote the binary forms of degree n (> 1), considered as a representation of* SL_2. *If the form $f \in V_n$ contains a linear factor of multiplicity one, then the orbit $O_f \subset V_n$ has the separation property.*

Proof. The case $n = 2$ is easy and is left to the reader. So we assume $n \geq 3$ from now on.

(a) We can assume that f has the form $f_0 := y(x + \alpha_2 y)(x + \alpha_3 y) \cdots (x + \alpha_n y)$. If $\lambda(t)$ denotes the diagonal matrix $\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} \in \mathrm{SL}_2$ then

$$t^{n-2} \cdot \lambda(t) f = y(x + t\alpha_2 y)(x + t\alpha_3 y) \cdots (x + t\alpha_n y) \to yx^{n-1} \text{ for } t \to 0.$$

This shows that $\overline{K^* O_f}$ contains the orbit of $x^{n-1}y$. As in the proof of Proposition 5 this allows to reduce to the case $f = x^{n-1}y$.

(b) Denote by L ($= V_1$) the linear binary forms and let $\mu\colon L \times L \to V_n$ be the morphism $(a, b) \mapsto ab^{n-1}$. This morphism is SL_2-equivariant, is bihomogeneous of degree $(1, n-1)$ and maps onto the orbit closure $\overline{O_{x^{n-1}y}}$. The comorphism $\mu^*\colon V_n^* \to L^* \otimes S^{n-1}L^* \subset \mathcal{O}(L \times L)$ is the embedding of the Cartan component $V_n^* \simeq S^n L^*$ of the CLEBSCH-GORDAN decomposition $L^* \otimes S^{n-1}L^* \simeq S^n L^* \oplus S^{n-2}L^*$. If $\ell \in V_n^*$ is a linear form and $H_\ell \subset V_n$ the corresponding hyperplane then $H_\ell \cap \overline{O_{x^{n-1}y}}$ is the image under μ of the zero set $\mathcal{V}_{L\times L}(\tilde{\ell})$ where $\tilde{\ell} := \mu^*(\ell) \in L^* \otimes S^{n-1}L^*$. Now $\tilde{\ell}$ is of the form $\tilde{\ell} = qm$ where $q \in \mathcal{O}(L \times L) = S(L^*) \otimes S(L^*)$ is irreducible of degree $(1, r)$ and $m \in \mathcal{O}(L) = S(L^*)$ is a product of linear forms, each of degree $(0, 1)$. Thus, the irreducible components of $H_\ell \cap \overline{O_{x^{n-1}y}}$ consists of the image of $\mathcal{V}_{L\times L}(q)$ under μ and the planes $a_j^{n-1}L \subset V_n$ where $\bigcup Ka_j \subset L$ is the zero set of $m \in \mathcal{O}(L)$. Therefore we see that if $\ell_1, \ell_2 \in V_n^*$ are two linear forms such that $H_{\ell_1} \cap \overline{O_{x^{n-1}y}} \subset H_{\ell_2}$, and $\ell_i = q_i m_i$ as above, then q_1 must be a scalar multiple of q_2. The following lemma implies that $\ell_1 = \ell_2$ (up to a scalar), hence the claim. □

Lemma 4. *Let $p = x \otimes p_1 + y \otimes p_2 \in L \otimes S^k L$ and $m_1, m_2 \in S^{m-k}L \setminus \{0\}$. If pm_1 and pm_2 both belong to the Cartan component $S^{m+1}L \subset L \otimes S^m L$ then pm_1 is a scalar multiple of pm_2.*

Proof. Recall that an element $q \in L \otimes S^m L$ belongs to the Cartan component if and only if $\Omega q = 0$ where

$$\Omega := \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial y} - \frac{\partial}{\partial y} \otimes \frac{\partial}{\partial x} \colon L \otimes S^m L \to S^{m-1}L$$

is the SL_2-equivariant projection operator. More precisely, we have

$$(*) \qquad L \otimes S^m L = \ker \Omega \oplus (x \otimes y - y \otimes x) S^{m-1}L$$

since $(x \otimes y - y \otimes x)$ is an invariant. If $q = x \otimes q_1 + y \otimes q_2$ then $\Omega q = \frac{\partial q_1}{\partial y} - \frac{\partial q_2}{\partial x}$.

Assume that m_1 and m_2 are linearly independent and put $h := \frac{m_2}{m_1}$, $q := pm_1$. Then, by assumption, q and qh both belong to the Cartan component $S^{k+r+1}L \subset L \otimes S^{k+r}L$, hence $\Omega q = \Omega(qh) = 0$. Moreover, h is of degree 0 and so $x\frac{\partial h}{\partial x} + y\frac{\partial h}{\partial y} = 0$.

Putting $q = x \otimes q_1 + y \otimes q_2$ we obtain $\Omega q = \frac{\partial q_1}{\partial y} - \frac{\partial q_2}{\partial x} = 0$. Now

$$0 = \Omega(qh) = \left(\frac{\partial q_1}{\partial y} - \frac{\partial q_2}{\partial x}\right) h + q_1 \frac{\partial h}{\partial y} - q_2 \frac{\partial h}{\partial x} = q_1 \frac{\partial h}{\partial y} - q_2 \frac{\partial h}{\partial x}$$

and so

$$\frac{q_1}{q_2} = \frac{\frac{\partial h}{\partial x}}{\frac{\partial h}{\partial y}} = -\frac{y}{x}.$$

Therefore, $xq_1 + yq_2 = 0$ which implies that $q_1 = yf$ and $q_2 = xf$ with a suitable element $f \in S^{m-1}L$. Hence $q = (x \otimes y - y \otimes x)f$. But $(x \otimes y - y \otimes x)f$ belongs to $S^{m-1}L \subset L \otimes S^m L$ (see $(*)$). This contractions proves the lemma. □

Remark 7. It is not hard to see that the orbit of the element $x^i y^{n-i} \in V_n$ does not have the separation property (SP) if i and $n-i$ are both different from 1. (In fact, let $f \in V_n$ be in the orbit of such an $x^i y^{n-i}$ and write $f = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \cdots + a_{n-1} x y^{n-1} + a_n y^n$. Then $a_0 = 0$ implies $a_1 = 0$ and so $H_{a_0} \cap O_{x^i y^{n-i}} \subset H_{a_1}$.)

§7. Separation property for families of subvarieties

The strong separation property (SSP) is an *open* property in the following sense. (We do not know if this is also the case for the other separation properties.)

Proposition 6. *Let $p\colon F \to S$ be a family if d-dimensional closed subvarieties of $\mathbb{P}^n$. Then the subset $\{s \in S \mid p^{-1}(s)$ satisfies (SSP)$\}$ is open in S.*

Recall that a *family of d-dimensional subvarieties* of $\mathbb{P}^n$ is a closed subvariety $F \subset S \times \mathbb{P}^n$, S an arbitrary variety, such that the projection pr_S induces a surjective morphism $p\colon F \to S$ with all fibers of dimension d. We put $F_s := p^{-1}(s)$. The next lemma immediately implies the proposition above.

Lemma 5. *Let $p\colon F \to S$ be a family if d-dimensional subspaces of $\mathbb{P}^n$, $n \geq d$. Fix an integer $1 \leq k \leq d$ and denote by $\mathcal{L}_k$ the set of linear subspaces $L \subset \mathbb{P}^n$ of codimension k. Then*

$$\{s \in S \mid \operatorname{codim}_{F_s} F_s \cap L = k \textit{ for all } L \in \mathcal{L}_k\}$$

is open in S.

Proof. Let $\mathrm{Gr} := \mathrm{Grass}_{n-k+1}(K^{n+1})$ the Grassmanian of $(n-k+1)$-dimensional subspaces of K^{n+1}, and let $q\colon B \to \mathrm{Gr}$ the corresponding subbundle of the trivial bundle $\mathrm{Gr} \times \mathbb{P}^n$, i.e. $q^{-1}([L]) = \{[L]\} \times L$. Define

$$Z := \{(y, b) \in F \times B \mid \mathrm{pr}_{\mathbb{P}^n}(y) = \mathrm{pr}_{\mathbb{P}^n}(b)\}.$$

Then Z is closed in $F \times B$ and the canonical projection $\varphi\colon Z \to S \times \mathrm{Gr}$ is proper. By definition, we have $\varphi^{-1}((s, [L])) \simeq F_s \cap L$. Therefore, φ is surjective and the subset $\{(s, [L]) \mid \operatorname{codim}_{F_s} F_s \cap L < k\}$ is closed in $S \times \mathrm{Gr}$. It follows that

$$\{s \in S \mid \operatorname{codim}_{F_s} F_s \cap L = k \text{ for all } L \in \mathcal{L}_k\}$$

is open in S, since $S \times \mathrm{Gr} \to S$ is a proper morphism. □

Appendix: Normality of orbit closures and multicones

We use the assumptions and the notation introduced in §4. In particular, G denotes a connected, semisimple and simply connected algebraic group defined over an algebraically closed field K of arbitrary characteristic. Let $V_i := V(\lambda_i)$ be Weyl modules of highest weights $\lambda_1, \lambda_2, \ldots, \lambda_s \in X^+$. Choose highest weight vectors $v_i \in (V_i)_{\lambda_i}$ and define

$$v := (v_1, v_2, \ldots, v_s) \in W := Kv_1 \oplus Kv_2 \oplus \cdots \oplus Kv_r \subset V := V_1 \oplus V_2 \oplus \cdots \oplus V_s$$

Proposition A. *GW is a closed normal subvariety of V with rational singularities.*

Proof. Let $P \subset G$ be the (reduced) normalizer of $W \subset V$ and let $\mathcal{V}$ be the associated vector bundle over G/P whose total space is $G \times^P W$. Then there is a canonical surjective and G-equivariant morphism

$$\varphi\colon G \times^P W \to Y := GW \quad \text{given by} \quad [g, w] \mapsto gw.$$

Hence, $\varphi^*\colon \mathcal{O}(Y) \hookrightarrow \mathcal{O}(G \times^P W)$ is an inclusion. We will show that φ^* is an isomorphism. Let $\mathcal{L}_i$ be the line bundle on G/P with total space $G \times^P (Kv_i)^*$. Then $H^0(G/P, \mathcal{L}_i) \simeq V_i^*$, and we obtain a canonical isomorphism

$$\mathcal{O}(G \times^P W) \xrightarrow{\sim} \bigoplus_{m \in \mathbb{N}^s} H^0(G/P, \mathcal{L}_1^{\otimes m_1} \otimes \mathcal{L}_2^{\otimes m_2} \otimes \cdots \otimes \mathcal{L}_s^{\otimes m_s})$$

of multigraded algebras. In this situation it is shown in [KeR87] that the algebra on the right hand side is normal with rational singularities (loc. cit. Theorem 2) and that it is generated by the linear part $\oplus_i H^0(G/P, \mathcal{L}_i) \xrightarrow{\sim} V_1^* \oplus V_2^* \oplus \cdots \oplus V_s^* = V^*$ (loc. cit. Theorem 3). Thus $\mathcal{O}(Y) \xrightarrow{\sim} \mathcal{O}(G \times^P W)$ and the claim follows. □

Remark 8. Assume that the highest weights $\lambda_1, \dots \lambda_r$ are linearly independent (over $\mathbb{Q}$). Then the orbit Pv is dense in W and so $Y = GW = \overline{Gv}$. Moreover, the orbits in Y are in one-to-one correspondence with the subsets of $I \subset \{1, 2, \dots, s\}$. In fact, define $v_I \in W$ by

$$(v_I)_i = \begin{cases} v_i & \text{if } i \in I, \\ 0 & \text{if } i \notin I. \end{cases}$$

and put $W_I := \bigcap_{j \notin I} \ker \mathrm{pr}_j = \bigoplus_{i \in I} V_i \subset W$. Then every orbit in Y is of the form Gv_I for some $I \subset \{1, 2, \dots, s\}$, and $\overline{Gv_I} = GW_I$.

Since $G_v \subset P$, the morphism $\varphi\colon G \times^P W \to Y$ is generically bijective. Similarly, $\varphi_I\colon G *_{P_I} W_I \to \overline{Gv_I} = GW_I$ is generically bijective for any subsets I where P_I is the normalizer of W_I. If $P_I \supsetneqq P$ for any strict subset I of $\{1, 2, \dots, s\}$, then $\operatorname{codim}_Y(Y \setminus Gv) \geq 2$ (since $W_I \subsetneqq W$ for any such subset I). This condition is satisfied if we have the following situation:

Define the *support* $\operatorname{supp}\lambda$ of λ to be the set of fundamental weights occurring in a representation of λ as a linear combination of fundamental weights. *If $\bigcup_{i \in I} \operatorname{supp}\lambda_i \subsetneqq \bigcup_j \operatorname{supp}\lambda_j$ for all strict subsets $I \supsetneqq \{1, 2, \dots, s\}$, then $P_I \supsetneqq P$.* This obviously holds if the λ_i's are (multiples of) fundamental weights.

Proposition B. *If the weights $\lambda_1, \lambda_2, \dots, \lambda_s$ are distinct fundamental weights and $v \in V_1 \oplus V_2 \oplus \cdots \oplus V_s$ is as above, then $\overline{Gv}$ is factorial (with rational singularities.)*

Proof. We use the notation introduced above. Since $Y = \overline{Gv}$ is normal we have to show that the divisor class group $\mathrm{Cl}(Y)$ is trivial.

(a) Set $\lambda = \sum_i \lambda_i$ and consider the morphism

$$\psi\colon V_1 \oplus V_2 \oplus \cdots \oplus V_s \to V_1 \otimes V_2 \otimes \cdots \otimes V_s$$

given in the obvious way. Then $\psi(W) = V(\lambda)_\lambda \subset V(\lambda)$. Therefore, the schematic stabilizer P of W is contained in the schematic stabilizer of $V(\lambda)_\lambda$ which is the reduced parabolic P_λ (Proposition 4(5)). Since $P_{\text{red}} = P_\lambda$ we see that P is reduced. We claim that P_v is reduced, too. In fact, P_v is the kernel of the homomorphism $\varphi\colon P \to K^{*s}$ given by the fundamental weights $\lambda_1, \ldots, \lambda_s$. Restricting φ to the maximal torus $T \subset P$ we see that the differential of φ is surjective. Moreover, $P = Z \times P_v$ where $Z \subset T$ is the center of P, i.e. the intersection of the kernels of the fundamental characters different from $\lambda_1, \ldots, \lambda_s$.

(b) The decomposition $P = Z \times P_v$ implies that the principal bundle $G \to G/P_v$ is locally trivial in the Zariski topology. Since G is simply connected it follows that the Picard group $\operatorname{Pic}(G/P_v)$ is isomorphic to the character group of P_v (see [FoI73, Remarks after Proposition 3.1], cf. [KKV89]). We claim that $X(P_v)$ is trivial. In fact, the character group of P is given by $X(P) = \bigoplus_{i=1}^s \mathbb{Z}\lambda_i$, because the λ_i's are different fundamental weights, and $P_v = \bigcap_{i=1}^s \ker \lambda_i$. Now it remains to remark that $\operatorname{Cl}(Y) = \operatorname{Cl}(Gv) = \operatorname{Pic}(G/P_v)$ (cf. [Har77, Proposition II.6.5(b) and Corollary II.6.16]), because the complement $Y \setminus Gv$ has at least codimension 2 (see Remark 8 above). □

REFERENCES

[Bau01] Baur, K., *Zerlegbare Tensoren und Cartankomponenten*, Doktorarbeit Basel, 2001.

[FoI73] Fossum, R.; Iversen, B., *On Picard groups of algebraic fibre spaces*, J. pure appl. Algebra **3** (1973), 269–280.

[Har70] Hartshorne, R., *Ample Subvarieties of Algebraic Varieties*, Lecture Notes in Math., vol. 156, Springer-Verlag, New York Heidelberg Berlin, 1970.

[Har77] ——, *Algebraic Geometry*, Graduate Texts in Math., vol. 52, Springer-Verlag, New York Heidelberg Berlin, 1977.

[His84] Hiss, G., *Die adjungierten Darstellungen der Chevalley-Gruppen*, Arch. Math. **42** (1984), 408–416.

[Jan87] Jantzen, J. C., *Representations of Algebraic Groups*, Pure and Applied Mathematics vol. 131, Academic Press, Inc., Boston–Orlando–San Diego–New York, 1987.

[Jan98] —— (Notes by I. Gordon), *Representations of Lie algebras in prime characteristic*, University of Aarhus, Preprint Series No. 1 (1998).

[KeR87] Kempf, G.; Ramanathan, A., *Multi-cones over Schubert varieties*, Invent. Math. **87** (1987), 353–363.

[KKV89] Knop, F.; Kraft, H.; Vust, Th., *The Picard group of a G-variety*, Algebraische Transformationsgruppen und Invariantentheorie (H. Kraft, P. Slodowy, T. A. Springer, eds.), DMV-Seminar, vol. **13**, Birkhäuser Verlag, Basel–Boston, 1989, pp. 77–87.

[KKL89] Knop, F.; Kraft, H.; Luna, D.; Vust, Th., *Local properties of algebraic group actions*, Algebraische Transformationsgruppen und Invariantentheorie (H. Kraft, P. Slodowy, T. A. Springer, eds.), DMV-Seminar, vol. **13**, Birkhäuser Verlag, Basel–Boston, 1989, pp. 63–75.

[Lau96] Lauritzen, N., *Embeddings of homogeneous spaces in prime characteristics*, Amer. J. Math. **118** (1996), 377–387.

[NeS99] Neubauer, M., Sethuraman, B.A., *Commuting pairs in the centralizers of 2-regular matrices*, J. Algebra **214** (1999), 174–181.

[Pre97] Premet, A., *Support varieties of non-restricted modules over Lie algebras of reductive groups*, J. London Math. Soc. (1997), 236–250.

[RaR85] Ramanan, S.; Ramanathan, A., *Projective normality of flag varieties and Schubert varieties*, Invent. Math. **79** (1985), 217–224.

[Ra85] Ramanathan, A., *Schubert varieties are arithmetically Cohen-Macaulay*, Invent. Math. **80** (1985), 283–294.

[ViP72] Vinberg, E., Popov, V.L., *On a class of quasihomogeneous affine varieties*, Math. USSR-Izv. **6** (1972), 743–758.

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