ON THE NULLCONE OF REPRESENTATIONS OF REDUCTIVE GROUPS

Hanspeter Kraft and Nolan R. Wallach

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Abstract. We study the geometry of the nullcone $N = N_{V^{\oplus k}}$ for several copies of a representation V of a reductive group G and its behavior for different k. We show that for large k there is a certain "stability" with respect to the irreducible components. In the case of the so-called θ-representations, this can be made more precise by using the combinatorics of the weight system as a subset of the root system. All this finally allows to calculate explicitly and in detail a number of important examples, e.g. the cases of 3- and 4-qubits which play a fundamental rôle in quantum computing.

Introduction

Let V be a finite dimensional complex representation of a complex reductive group G and denote by $N_V \subset V$ the *nullcone*, i.e. the zero set of the homogeneous non-constant invariant functions on V. This cone was introduced and studied by Hilbert in his famous paper "Über die vollen Invariantensysteme" [Hi93] where he gives a "constructive" proof of the finiteness theorem for the invariants.

The nullcone N_V plays a fundamental rôle in the geometry of the representation, in particular, in problems concerning the structure of orbits and their closure and in the study of the algebraic quotient $\pi_V: V \to V/\!\!/G$ given by the invariants. E.g., if $f_1, f_2, \dots, f_r$ are algebraically independent homogeneous functions defining the nullcone N_V then the invariants $\mathbb{C}[V]^G$ form a free module over the polynomial ring $\mathbb{C}[f_1, f_2, \dots, f_r]$. Moreover, knowing the degrees of the f_i's one can immediately give an upper bound for the degrees of a generating system for the invariants. Moreover, there are efficient tools to calculate the Hilbert series of the invariants. We refer the reader to [DeKr97] and to the book [DeKe02] for more details.

In trying to understand the geometry of the n-qubits, i.e. the representation $Q_n := \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$ (n factors) of $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \cdots \times \mathrm{SL}_2$ (also n factors, with obvious action on Q_n) we became interested in the structure of the nullcone $N_{V^{\oplus k}}$ for k copies of a given representation V and in particular in its behavior for large k. It turned out that there is a "stability" property saying that the general structure does not change anymore once k is greater than a certain number $m(V) <$

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$\dim V$ calculated from the weight system of V (see §1 Theorem 1 for a precise statement). E.g. for the 3-qubits Q_3 we find that *the nullcones* N_{Q_3} *and* $N_{Q_3^{\oplus 2}}$ *are both irreducible, of dimension 7 and 11, respectively. The nullcone of* $k \geq 3$ *copies has 4 components, one of dimension* $3+4k$ *which is stable under* S_3 *and three of dimension* $1+4k$ *which are permuted by* S_3 (see §2 Example 4).

On the other hand, the nullcone of any number of copies of the adjoint representation of a semisimple group G on its Lie algebra $\mathfrak{g}$ is irreducible and has a nice resolution of singularities, namely $G \times_B \mathfrak{n}^{\oplus k} \to N_{\mathfrak{g}^k}$ where $B \subset G$ is a Borel subgroup and $\mathfrak{n} := \operatorname{Lie} B_u$ is the Lie algebra of the unipotent radical of B. A similar behavior can be found for all representations of SL_2 where the nullcone for any number of copies is irreducible. But this behavior is rather exceptional for groups of rank ≥ 2 and seems to occur only for "small" representations, e.g. for the standard representation of SL_n (and its dual) and for the 7-dimensional representation of G_2. We do not know a classification of these representations.

The next type of examples we were studying are the so-called θ-representations introduced by Vinberg [Vi76], motivated by and generalizing the situation of symmetric spaces studied by Kostant and Rallis [KoR71] (see §3). One of the reasons for us was the fact that the 4-qubits Q_4 are a θ-representation (see §4). In these cases we are able to prove more precise results (see §3 Proposition 4 and Corollary 1) which enabled us to determine the nullcone for many important examples. E.g. in the case of 4-qubits we find that N_{Q_4} *is irreducible where as for* $k \geq 2$ *the nullcone* $N_{Q_4^{\oplus k}}$ *has 12 irreducible components, decomposing into 3 orbits of 4 elements under* S_4*. The dimensions are* $8k+4$*,* $8k+3$ *and* $8k+1$ (§4 Proposition 7).

As already said above one of the main purposes of this paper are the examples worked out in detail which show a very interesting behavior of the nullcone, mostly irregular and sometimes quite surprising. Although we do not have a complete picture or a final answer—maybe there is none—we believe that the general results explain some of the phenomena and that the examples will help to get a deeper insight into the situation.

After finishing this paper we were informed about a paper of V.L. Popov [Po03] where he gives a general algorithm to determine the irreducible components of maximal dimension of the nullcone, using the weights of the representation and their multiplicities.

§1. Irreducible components of the nullcone

Our base field is the field $\mathbb{C}$ of complex numbers. Let G be a connected reductive group and $\rho\colon G \to \mathrm{GL}(V)$ a finite dimensional (rational) representation. The *nullcone* N_V of the representations V is defined by

$$N_V := \{v \in V \mid \overline{Gv} \ni 0\} = \pi_V^{-1}(\pi_V(0)),$$

where $\pi_V\colon V \to V/\!\!/G$ is the *quotient* morphism (see [Kr85, Kap. II] or [MFK94]). The nullcone plays a fundamental rôle in the study of the geometry of the representation V and of the quotient morphism.

In this section we will describe the irreducible components of $N_{V^{\oplus k}}$ for the representation of G on $V^{\oplus k} = V \oplus V \oplus V \cdots \oplus V$ (k copies of V) and show how $N_{V^{\oplus k}}$ behaves for $k \to \infty$.

By the Hilbert-Mumford criterion we know that a vector $v \in V$ belongs to the nullcone N_V if and only if there is a one-parameter subgroup (short: 1-PSG) $\lambda\colon \mathbb{C}^* \to G$ such that $\lim_{t\to 0} \lambda(t)v = 0$ (loc. cit. Kap. III.2). For a given 1-PSG $\lambda\colon \mathbb{C}^* \to G$ we define

$$V(\lambda) := \{v \in V \mid \lim_{t\to 0} \lambda(t)v = 0\}.$$

Let $T \subset G$ be a maximal torus and denote by $X(T)$ the character group of T. Then we have the following *weight decomposition*

$$V = \bigoplus_{\gamma\in X(T)} V_\gamma, \quad V_\gamma := \{v \in V \mid \rho(t)v = \gamma(t)\cdot v \text{ for } t \in T\}.$$

V_γ is called the *weight space* of weight γ. If $\lambda\colon \mathbb{C}^* \to T \subset G$ a 1-PSG of T, then

$$V(\lambda) = \bigoplus_{\langle\lambda,\gamma\rangle>0} V_\gamma$$

where $\langle\lambda, \gamma\rangle$ denotes the usual pairing between $X(T)$ and the group $Y(T)$ of 1-PSG's of T: $\langle\lambda, \gamma\rangle = n$ if $\gamma(\lambda(t)) = t^n$ for $t \in \mathbb{C}^*$.

Varying $\lambda \in Y(T)$ we find finitely many different subspaces $V(\lambda)$ of V. Using the Hilbert-Mumford criterion mentioned above and the fact that every 1-PSG of G is conjugate to a 1-PSG of T, we obtain the following description of the nullcone of $V^{\oplus k}$ for any $k \geq 1$:

$$N_{V^{\oplus k}} = \bigcup_{\lambda\in Y(T)} G\,V(\lambda)^{\oplus k}.$$

Moreover, $V(\lambda)$ is normalized by a parabolic subgroup $P(\lambda)$ containing T which depends on λ and on V. In fact,

$$P(\lambda) \supseteq \{g \in G \mid \lim_{t\to 0} \lambda(t)g\lambda(t)^{-1} \text{ exists}\},$$

and the right hand side is a parabolic subgroup with Levi subgroup G^λ (see [Ke78, page 305]. Clearly, we have equality for a suitably chosen λ. It follows that $G\,V(\lambda)$ is closed and that there is a proper surjective morphism

$$G \times_{P(\lambda)} V(\lambda)^{\oplus k} \to G\,V(\lambda)^{\oplus k}$$

where $G \times_{P(\lambda)} V(\lambda)^{\oplus k}$ denotes the associated vector bundle to the principal bundle $G \to G/P$, i.e., $G \times_{P(\lambda)} V(\lambda)^{\oplus k} := (G \times V(\lambda)^{\oplus k})/P(\lambda)$ with the diagonal action of $P(\lambda)$ given by $p(g, v) := (gp^{-1}, pv)$.

Let us denote by $X_V \subset X(T)$ the set of weights of V. Given any subset $X \subset X_V$ we put $V_X := \oplus_{\gamma\in X} V_\gamma$.

Definition 1. A subset $X \subset \mathcal{X}_V$ is called *unstable* if there is a $\lambda \in Y(T)$ such that $\langle \lambda, \alpha \rangle > 0$ for all $\alpha \in X$, and *maximal unstable* if X is maximal under this condition. We denote by Ξ_V the set of maximal unstable subsets of $\mathcal{X}_V$.

Clearly, we have

$$(*) \qquad N_{V^{\oplus k}} = \bigcup_{X \in \Xi_V} G\, V_X^{\oplus k}.$$

The Weyl group $W = N_G(T)/T$ acts on $\mathcal{X}(T)$ and leaves $\mathcal{X}_V$ invariant. If $X \subset \mathcal{X}_V$ is maximal unstable then so is wX, and $GV_X = GV_{wX}$. Thus it suffices in $(*)$ to choose a representative for any W-orbit of maximal unstable subsets of $\mathcal{X}_V$. Moreover, this representative can be chosen in such a way that V_X is stable under a fixed Borel subgroup B of G.

Definition 2. A subspace $U \subset V$ is called *unstable* if U is annihilated by a 1-PSG λ (i.e. $U \subset V(\lambda)$), and *maximal unstable* if it is maximal under this condition. We denote by $\mathcal{U}_V$ the set of maximal unstable subspaces of V.

If $X \subset \mathcal{X}_V$ is a maximal unstable subset then V_X is a maximal unstable subspace. Conversely, if $U \subset V$ is a maximal unstable subspace, then $U = gV_X$ for a suitable $g \in G$ and a maximal unstable subset $X \subset \mathcal{X}_V$. Moreover, if U is T-stable, then $U = V_X$ for some $X \in \Xi_V$.

We will consider $\mathcal{U}_V$ as a G-stable subset of the Grassmannian:

$$\mathcal{U}_V \subset \mathrm{Gr}(V) := \bigcup_{1 \le m \le \dim V} \mathrm{Gr}_m(V).$$

In fact, $\mathcal{U}_V$ consists of finitely many closed orbits since each maximal unstable subset U is normalized by a parabolic subgroup.

In order to determine which spaces $V_X^{\oplus k}$ contribute to irreducible components of $N_{V^{\oplus k}}$ in the decomposition $(*)$ we have to see if $V_{X'}^{\oplus k} \subset GV_X^{\oplus k}$. The next lemma is a first step in this direction.

Lemma 1. *Let $U \subset V$ be a B-stable subspace where $B \subset G$ is a Borel subgroup containing T. Assume that there is a $g \in G$ and a subset $X' \subset \mathcal{X}(T)$ such that $gU \subset \bigoplus_{\gamma \in X'} V_\gamma$. Then $w\mathcal{X}_U \subset X'$ for a suitable $w \in W$.*

Proof. We can write g in the form $g = un_w b$ where $b \in B$, $n_w \in N_G(T)$ is a representative of some $w \in W$, and $u \in B_u$, the unipotent radical of B. Then $bU = U$ and so $U_1 := n_w bU$ is T-stable with weights $\mathcal{X}_{U_1} = w\mathcal{X}_U$. It follows that the subspace $gU = uU_1$ has the property that its projection onto every weight space $V_\gamma, \gamma \in w\mathcal{X}_U$, is non-zero. On the other hand, we have $uU_1 = gU \subset \bigoplus_{\gamma \in X'} V_\gamma$, by assumption. This implies that $w\mathcal{X}_U \subset X'$. □

For any representation V of G we define

$$m(V) := \max\{\dim U \mid U \subset V \text{ maximal unstable subspace}\}.$$

We always have $m(V) < \dim V$. If V is selfdual, i.e. $\mathcal{X}_V = -\mathcal{X}_V$, then

$$m(V) = \frac{\dim V - \dim V_0}{2}.$$

In fact, if X is a maximal unstable subset of $\mathcal{X}_V$ and if γ is a non-zero weight of V then exactly one of $\gamma, -\gamma$ is contained in X.

Theorem 1. *Let V be a representation of G and let $\{X_1, X_2, \dots, X_s\}$ be a set of representatives of the W-orbits of maximal unstable subsets of X_V. If every irreducible subrepresentation W of V occurs at least $m(W)$ times, then the closed subsets $C_i := G\,V_{X_i}$ are the distinct irreducible components of the nullcone N_V. Moreover, the canonical morphism $G \times_{P_i} V_{X_i} \to C_i$ where P_i is the normalizer of V_{X_i} is a resolution of singularities, i.e., it is proper and birational.*

Proof of Theorem 1. Assume that $V_{X_i} \subset G\,V_{X_j}$ for some $i \neq j$. Fix a decomposition of V into irreducible factors:

$$\varphi\colon V \xrightarrow{\sim} \bigoplus_{\gamma} W_\gamma^{\oplus m_\gamma}$$

where the W_γ are pairwise non-isomorphic simple G-modules. Then φ induces isomorphisms $\varphi_i\colon V_{X_i} \xrightarrow{\sim} \bigoplus_\gamma m_\gamma W_{\gamma X_i}$ for all i. Since $m_\gamma \geq \dim W_{\gamma X_i}$ we can find a vector $v \in V_{X_i}$ such that the components of $\varphi(v)$ in the m_γ copies of W_γ span the subspace $W_{\gamma X_i} \subset W_\gamma$ for every γ. By assumption, there is a $g \in G$ such that $gv \in V_{X_j}$. It follows that $gW_{\gamma X_i} \subset W_{\gamma X_j}$ for all γ, and so $gV_{X_i} \subset V_{X_j}$. Now we can apply Lemma 1 above to find a $w \in W$ such that $wX_i \subset X_j$. This contradiction proves the first part of the theorem.

For the second part we have already remarked that $P_i \subset G$ is a parabolic subgroup, that $C_i \subset N_V$ is closed and that the canonical morphism

$$\eta_i\colon G \times_{P_i} V_{X_i} \to C_i, \quad [g, v] \mapsto gv$$

is proper and surjective. Now we use the same decomposition $\varphi\colon V \xrightarrow{\sim} \bigoplus_\gamma W_\gamma^{m_\gamma}$ as above. It is easy to see that the vectors $v \in V_{X_i}$ with the property that the components of $\varphi(v)$ in the m_γ copies of W_γ span the subspace $W_{\gamma X_i}$ for all γ form a dense open subset $Z_i \subset V_{X_i}$. For any $v \in Z_i$ we therefore have $\operatorname{Stab}_G v = \operatorname{Stab}_G V_{X_i} \subset \operatorname{Norm}_G V_{X_i} = P_i$. Thus, η_i is injective on the dense subset $G \times_{P_i} Z_i$ of $G \times_{P_i} V_{X_i}$, hence birational. $\square$

Remark 1. Let $\Delta \subset X(T)$ be the root system, $B \supset T$ a Borel subgroup of G and $\Delta^+ \subset \Delta$ the corresponding positive roots. In order to determine the W-representatives $\{X_1, X_2, \dots, X_s\}$ of the maximal unstable subsets it suffices to consider those subsets $X \subset X(T)$ where the corresponding subspace V_X is invariant under B. This means that X satisfies the condition

$$(X + \Delta_+) \cap X_V \subset X.$$

We will call these subsets Δ^+-*invariant* or B-*invariant*.

Remark 2. If $p\colon V \to V'$ is a surjective G-homomorphism then the induced morphism $p\colon N_V \to N_{V'}$ is surjective. In particular, $N_{V'} = \bigcup_i p(C_i)$ and the images $p(C_i)$ are all closed, but there might be some inclusions between the different $p(C_i)$.

The next result is an immediate consequence of the proof of Theorem 1 above. There is an obvious generalization to arbitrary representations V whose formulation is left to the reader.

Theorem 2. *Let V be an irreducible representation of G and let $\{X_1, \dots, X_s\}$ be representatives of the W-orbits of maximal unstable subsets of X_V. For a given $k \geq 1$ the corresponding component $G\, V_{X_i}^{\oplus k} \subset N_{V^{\oplus k}}$ is not visible (i.e. is contained in another component $G\, V_{X_j}^{\oplus k}$) if and only if the following condition holds:*

(C_k) *For every k-dimensional subspace $U \subset V_{X_i}$ there is an element $g \in G$ and an index $j \neq i$ such that $gU \subset V_{X_j}$.*

§2. Examples

In this section we give a number of important examples which were the motivation for the study of the nullcone of multiple copies of a given representation.

Example 1: Adjoint representations. Let $\mathfrak{g} := \operatorname{Lie} G$ be the Lie algebra of G. Then $N_{\mathfrak{g}}$ is the set of nilpotent elements in $\mathfrak{g}$. Fixing a Borel subgroup B of G we see that there is a unique maximal B-invariant unstable subspace in $\mathfrak{g}$, namely the nilradical $\mathfrak{n}$ of $\operatorname{Lie} B$. The corresponding $X_{\mathfrak{n}}$ is the set of positive roots. This implies the following:

For any $k \geq 1$ the nullcone $N_{\mathfrak{g}^{\oplus k}}$ is irreducible and

$$G \times_B \mathfrak{n}^{\oplus k} \to N_{\mathfrak{g}^{\oplus k}}$$

is a resolution of singularities. Moreover, $\dim N_{\mathfrak{g}^{\oplus k}} = (k+1)\frac{\dim G - \operatorname{rk} G}{2}$.

Example 2: Orthogonal representations. Let $G := \mathrm{SO}_n$ be the special orthogonal group and $V := \mathbb{C}^n$ the standard representation. We claim that for odd n there is a unique maximal B-invariant unstable subset whereas for even n there are two. More precisely, we have the following result.

Let V be the standard representation of the orthogonal group SO_n.

1. *If $n = 2m+1 \geq 3$ is odd then $N_{V^{\oplus k}}$ is irreducible for any number k of copies of V.*
2. *If $n = 2m \geq 2$ is even then $N_{V^{\oplus k}}$ is irreducible for $k < m$ and has two irreducible components for $k \geq m$, permuted by O_n.*

Moreover, in both cases $\dim N_V = n-1$ and $\dim N_{V^{\oplus k}} \leq \binom{m}{2} + km$ with equality for $k \geq m$.

Proof. (1) For $n = 2m+1$ the weights of V are $\{\pm\varepsilon_1, \pm\varepsilon_2, \dots, \pm\varepsilon_m, 0\}$. Since the positive roots Δ^+ contain the elements $\{\varepsilon_i - \varepsilon_j \mid i < j\}$ and $\{\varepsilon_k \mid 1 \leq k \leq m\}$ it easily follows that there is exactly one maximal Δ^+-invariant unstable subset, namely $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m\}$.

(2) For $n = 2m$ the weights are $\{\pm\varepsilon_1, \pm\varepsilon_2, \dots, \pm\varepsilon_m\}$ and the positive roots Δ^+ contain the elements $\{\varepsilon_i \pm \varepsilon_j \mid i < j\}$. It follows that there are two maximal Δ^+-invariant unstable subsets, $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m\}$ and $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m-1}, -\varepsilon_m\}$, which are not equivalent under the Weyl group. The corresponding subspaces U and U' are both totally isotropic and are equivalent under O_n, but not under SO_n. On the other hand, $\mathrm{GL}(U)$ can be identified with a Levi subgroup of SO_n and so every

linear automorphism of U (or U') is induced by an element of SO_n. Hence, for every linear subspaces W of U' of dimension $< m$ there is an element $g \in \mathrm{SO}_n$ such that $gW \subset U \cap U' \subset U$. This proves the claim by Theorem 2 above.

The statement about dimensions follows from the fact that the normalizer of U is the maximal parabolic subgroup P with Levi factor GL_m whose codimension in SO_n is $\binom{m}{2}$. □

Example 3: Quadratic forms. (This example has been worked out by Matthias Bürgin [Bür04].) Let $G = \mathrm{SL}_3$ with positive roots $\Delta^+ = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3, \varepsilon_2 - \varepsilon_3\}$, acting linearly on $V := S^2(\mathbb{C}^3)^*$, the quadratic forms in 3 variables.

There are two maximal Δ^+-invariant unstable subsets, and the corresponding two components of the nullcone are already seen in 2 copies $V \oplus V$. The nullcone N_V is an irreducible hypersurface of dimension 5, and, for $k \geq 2$, $N_{V^{\oplus k}}$ is equidimensional with two components of dimension $2 + 3k$.

Proof. The weights of V are $\{\varepsilon_i + \varepsilon_j \mid 1 \leq i \leq j \leq 3\}$ and one easily finds the following two maximal Δ^+-invariant unstable subsets: $X_1 := \{2\varepsilon_1, 2\varepsilon_2, \varepsilon_1 + \varepsilon_2\}$ and $X_2 := \{2\varepsilon_1, \varepsilon_1 + \varepsilon_2, \varepsilon_1 + \varepsilon_3\}$. Note that X_2 is W-equivalent to the complement $X_V \setminus X_1$. The corresponding subspaces of V are $V_1 := \langle x_1^2, x_1x_2, x_2^2\rangle$ and $V_2 := x_1 \cdot \langle x_1, x_2, x_3\rangle$. It follows that the 2-dimensional subspace $\langle x_1^2, x_2^2\rangle \subset V_1$ is not equivalent to a subspace of V_2, because it contains two linearly independent squares. Similarly, one sees that the subspace $\langle x_1x_2, x_1x_3\rangle \subset V_2$ is not equivalent to a subspace of V_1. Thus both components are already seen in two copies (Theorem 2).

The normalizers of V_1 and V_2 are the two parabolic subgroups P_1 and P_2 of codimension 2. Since the nullcone of one copy has codimension 1, hence dimension $5 = 2 + 3$ this implies that for all $k \geq 1$ the morphisms $G \times^{P_i} V_i^{\oplus k} \to G\, V_i^{\oplus k}$ are of finite degree and so $\dim N_{V^{\oplus k}} = 2 + 3k$. □

Example 4: The case of 3-qubits. Let $G := \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$ and $V := \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ with the usual linear action of G. Then the positive roots are given by $\Delta^+ = \{\alpha = 2\varepsilon_1, \alpha' = 2\varepsilon_1', \alpha'' = 2\varepsilon''\}$.

There are four maximal Δ^+-invariant unstable subsets, three of them are permuted by the obvious action of S_3 on V and hence on X_V. The nullcones N_V and $N_{V^{\oplus 2}}$ are both irreducible, of dimension 7 and 11, respectively. The nullcone of $k \geq 3$ copies has 4 components, one of dimension $3 + 4k$ which is stable under S_3 and three of dimension $1 + 4k$ which are permuted by S_3.

Proof. Setting $\mathbb{C}^2 = \mathbb{C}e_0 \oplus \mathbb{C}e_1$ we get the basis $(e_{ijk} := e_i \otimes e_j \otimes e_k \mid i, j, k \in \{0, 1\})$ of V. Since $\varepsilon_1 = -\varepsilon_0$ for the weights on $\mathbb{C}^2$ the corresponding weights ε_{ijk} of V are given by the vertices of a cube. It is easy to see that the maximal Δ^+-invariant unstable subsets of $X_V = \{\varepsilon_{ijk}\}$ are

$$X_0 := \{\varepsilon_{000}, \varepsilon_{001}, \varepsilon_{010}, \varepsilon_{100}\},$$
$$X_1 := \{\varepsilon_{000}, \varepsilon_{001}, \varepsilon_{010}, \varepsilon_{011}\},$$
$$X_2 := \{\varepsilon_{000}, \varepsilon_{001}, \varepsilon_{100}, \varepsilon_{101}\},$$
$$X_3 := \{\varepsilon_{000}, \varepsilon_{010}, \varepsilon_{100}, \varepsilon_{110}\}.$$

X_0 consists of the vertices of all edges containing ε_{000} and X_1, X_2, X_3 correspond to the 3 faces with vertex ε_{000}. The latter are permuted by the action of S_3 on the weights ε_{ijk} given by permuting the indices, and X_0 is invariant under S_3.

The normalizer of V_{X_0} is the Borel subgroup $B \times B \times B$ whereas $V_{X_1} = e_0 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ is normalized by $B \times \mathrm{SL}_2 \times \mathrm{SL}_2$, and similarly for V_{X_2} and V_{X_3}. The claim now follows from the following Lemma 2 together with Theorem 2. The statements about dimensions follow in a similar way as in Example 3. □

Lemma 2.

(1) *Every 2-dimensional subspace U of V_{X_1} can be mapped into $V_{X_1} \cap V_{X_0}$ with an element of $B \times \mathrm{SL}_2 \times \mathrm{SL}_2$.*

(2) *A generic 3-dimensional subspace of V_{X_1} cannot be mapped into V_{X_0} with an element of $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$.*

Proof. (1) Recall that the representation of $\mathrm{SL}_2 \times \mathrm{SL}_2$ on $V_{X_1} \simeq \mathbb{C}^2 \otimes \mathbb{C}^2$ corresponds to the standard representation of SO_4 on $\mathbb{C}^4$ where the invariant form on $\mathbb{C}^2 \otimes \mathbb{C}^2$ is given by

$$(e_{ij}, e_{i'j'}) = \begin{cases} 1 \text{ if } i+i' = j+j' = 1, \\ 0 \text{ otherwise.} \end{cases}$$

Thus every 2-dimensional subspace of V_{X_1} is perpendicular to an isotropic vector. Since SO_4 acts transitively on the isotropic vectors this implies that every 2-dimensional subspace of V_{X_1} can be mapped into the subspace perpendicular to the highest weight vector corresponding to ε_{000} which is $V_{X_1} \cap V_{X_0} = e_0 \otimes \langle e_{00}, e_{01}, e_{10} \rangle$.

(2) Let $U \subset V_{X_1}$ be any 3-dimensional subspace which is orthogonal to a non-isotropic vector. Then $hU \neq V_{X_1} \cap V_{X_0}$ for any $h \in \mathrm{SO}_4$ and so hU always contains a vector whose coordinate in the weight space of $\mathbb{C}e_{011}$ is non-zero. But this implies that for any $g \in G$ the image gU contains a vector whose projection into the subspace $V_{\{\varepsilon_{011}, \varepsilon_{111}\}} \subset V$ is non-zero. Thus $gU \not\subseteq V_{X_0}$ for all $g \in G$. □

§3. Nullcones for θ-representations

Let V be a representation of a reductive group G, and let $K \subset G$ a reductive subgroup and $W \subset V$ an K-stable subspace. Then we have an induced morphism $W/\!\!/K \to V/\!\!/G$. *This morphism is finite if and only if we have the following relation between the nullcone N_W of W with respect to K and the nullcone N_V of V with respect to G:*

$$N_W = N_V \cap W.$$

Equivalently, the restrictions of the G-invariants on V define a subalgebra $A \subset \mathcal{O}(W)^K$ such that $\mathcal{O}(W)^K$ is a finitely generated A-module. There are well-known examples where this holds.

Examples.

(1) Let G be semisimple, $\mathfrak{g} = \operatorname{Lie} G$ the adjoint representation, $K := \operatorname{Norm}_G T$ the normalizer of a maximal torus T in G and $\mathfrak{h} := \operatorname{Lie} T$; then, by Chevalley's restriction theorem, $\mathfrak{h}/\!\!/K \xrightarrow{\sim} \mathfrak{g}/\!\!/G$, i.e. $\mathcal{O}(\mathfrak{g})^G = \mathcal{O}(\mathfrak{h})^K$.

(2) Let $Gv = \overline{Gv} \subset V$ be a closed orbit and denote by G_v the stabilizer of v. Define $K := \operatorname{Norm}_G G_v$ and set $W := V^{G_v}$. Then the induced morphism $W/\!\!/K \to V/\!\!/G$ is a closed immersion. In particular, $\mathcal{O}(V)^G|_W = \mathcal{O}(W)^K$, i.e. $\mathcal{O}(W)^K$ is generated by the restrictions of the G-invariants of V to W. This generalization of (1) is due to Luna-Richardson [LuR79].

(3) Let G be a semisimple group and θ an automorphism of finite order of G. Define $G^\theta := \{g \in G \mid \theta(g) = g\}$. The automorphism θ defines an automorphism of $\mathfrak{g} := \operatorname{Lie} G$, also denoted by θ. Let $W \subset \mathfrak{g}$ be an eigenspace of θ and consider W as a representation of G^θ. Then $N_{\mathfrak{g}} \cap W = N_W$, see [Vi75], [Vi76].

Question 1. If such a "restriction property" holds for the pair $(G, V) \supset (K, W)$, does it also hold for two or more copies of V and W, i.e. do we have

$$(*) \qquad N_{V^{\oplus m}} \cap W^{\oplus m} = N_{W^{\oplus m}} \quad \text{for } m \geq 2?$$

In the above examples this is true for (1). We even have that $\mathcal{O}(\mathfrak{g}^n)^G|_{\mathfrak{h}^n} = \mathcal{O}(\mathfrak{h}^n)^K$, but this is a difficult theorem due to Joseph ([Jo97], cf. [Wa93]). This stronger result does not carry over to the generalization (2), but it might still be true that $(*)$ holds. Concerning the last example we will now show that $(*)$ holds in a more general situation.

Theorem 3. *Let $\rho: G \to \mathrm{GL}(V)$ be a representation of a reductive group G and let θ be a semisimple automorphism of V which normalizes $\rho(G)$. Consider the subgroup $K := \{h \in G \mid \rho(h)\theta = \theta\rho(h)\}$, and let $W \subset V$ be an eigenspace of θ. Then K is reductive and $N_V \cap W = N_W$. In particular, the induced morphism $W/\!\!/K \to V/\!\!/G$ is finite.*

Proof. It is well-known that K is reductive. A short argument can be found in [KSW99, Corollary 2] where we show that a closed subgroup $K \subset G$ of a reductive group G is reductive if and only if $\operatorname{Lie} H$ has an $\operatorname{Ad}(H)$-stable complement in $\operatorname{Lie} G$.

Since $N_W \subset N_V$ it remains to show that $N_V \cap W \subseteq N_W$. Multiplying θ with a suitable scalar we can assume that $W = V^\theta$. Define $S := \overline{\langle\theta\rangle} \subset \mathrm{GL}(V)$, the closure of the subgroup generated by θ. This group S is commutative and reductive, normalizes $\rho(G)$ and acts trivially on W. Therefore, $\tilde{G} := S \cdot \rho(G) \subset \mathrm{GL}(V)$ is again reductive. Since $\tilde{K} := \operatorname{Zent}_{\tilde{G}}(S)$ has the same image in $\mathrm{GL}(W)$ as K and since the nullcone of V with respect to $\tilde{G}$ contains N_V it suffices to prove the claim with G replaced by $\tilde{G}$ and K replaced by $\tilde{K}$. But now we are in the situation of Corollary 4.5 of [Ke78] which implies that for every point in $x \in N_V \cap W$ there is a 1-PSG λ of $\tilde{K}$ such that $\lim_{t\to 0} \lambda(t)x = 0$. □

From now on we will consider the following special case which was studied in detail by Vinberg in [Vi76]. Let G be a connected reductive group, $\mathfrak{g} := \operatorname{Lie} G$ its

Lie algebra, θ a semisimple automorphism of G and $K := G^\theta$ the fixed point group. We know that K is reductive, but not necessarily connected. The automorphism θ induces an automorphism of the Lie algebra $\mathfrak{g}$, also denoted by θ, and $\mathfrak{g}^\theta = \mathfrak{k} := \operatorname{Lie} K$. Moreover, every eigenspace $V \subset \mathfrak{g}$ of θ is a representation of K. We fix a θ-stable Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ such that $\mathfrak{t}_0 := \mathfrak{t} \cap \mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{k}$ and denote the corresponding maximal tori by $T \subset G$ and $T_0 \subset K$.

If $\mathfrak{b} \subset \mathfrak{g}$ is a θ-stable Borel subalgebra, then the intersection of the nilradical $\mathfrak{n}_\mathfrak{b}$ of $\mathfrak{b}$ with an eigenspace V of θ is an unstable subspace of V. We will see that all maximal unstable subspaces are obtained in this way.

Fix a Borel subalgebra $\mathfrak{b}_0 \subset \mathfrak{k}$ containing $\mathfrak{t}_0$ and define the following set of unstable subspaces:

$$\mathcal{B}_V := \{V_\mathfrak{b} := \mathfrak{n}_\mathfrak{b} \cap V \mid \mathfrak{b} \text{ a } \theta\text{-stable Borel subalgebra of } \mathfrak{g} \text{ containing } \mathfrak{b}_0 \text{ and } \mathfrak{t}\}.$$

Proposition 1. *Let $V \subset \mathfrak{g}$ be an eigenspace of θ. Then the maximal elements of $\mathcal{B}_V$, $\mathcal{B}_V^{\max}$, form a set of representatives of the K^0-equivalence classes of maximal unstable subspaces of V. In particular, for any number n of copies of V we have*

$$N_{V^{\oplus n}} = \bigcup_{U \in \mathcal{B}_V^{\max}} K^0\, U^{\oplus n}.$$

Proof. If $U \subset V$ is an unstable subspace annihilated by the 1-PSG λ of T_0 we can assume that λ is regular as a 1-PSG of T. This follows easily from the fact that $\mathfrak{t}^{\text{reg}} \cap \mathfrak{t}_0$ is open and dense in $\mathfrak{t}_0$ (see [BoM55]). Then $\mathfrak{n} := \{x \in \mathfrak{g} \mid \lim_{t\to 0} \lambda(t)x = 0\}$ is the nilpotent radical of a θ-stable Borel subalgebra $\mathfrak{b}$, and $\mathfrak{n} \cap V$ is unstable and contains U, by construction. By conjugation with a suitable element of K^0 we can assume that $\mathfrak{b}$ contains $\mathfrak{b}_0$. Thus every maximal unstable subspace is conjugate to a maximal element from $\mathcal{B}_V$.

Let $B_0 \subset K$ be the Borel subgroup with Lie algebra $\mathfrak{b}_0$. If two maximal subspaces $U_1, U_2 \in \mathcal{B}_V$ are equivalent under K^0 then there is an element $n \in \operatorname{Norm}_{K^0}(T_0)$ which sends U_1 onto U_2. This implies that U_2 is stable under B_0 and nB_0n^{-1}. Hence the normalizer of U_2 contains the subgroup H generated by B_0 and nB_0n^{-1}. Since H contains n we get $U_1 = U_2$. □

Question 2. Which maximal unstable subspaces from $\mathcal{B}_V$ contribute to irreducible components of N_V, and how many components do we have?

One knows that N_V is a complete intersection and contains only finitely many K-orbits. More precisely, the invariant ring of V is generated by algebraically independent elements, the quotient morphism $\pi\colon V \to V/\!\!/K$ is flat and each fiber of π contains only finitely many orbits (see [Vi76]).

Since θ normalizes $\mathfrak{t}$ we have an action of θ on the Weyl group $W := W(G,T)$. Denote by $W^\theta \subset W$ the subgroup of elements fixed by θ. It is easy to see that W^θ acts simply transitively on the set of θ-stable Borel subalgebra containing $\mathfrak{t}$. Therefore, the number of θ-stable Borel subalgebra containing $\mathfrak{t}$ and $\mathfrak{b}_0$ is given by the index $[W^\theta : W_K]$ where $W_K := W(K^0, T_0)$. (Recall that the restriction of W^θ to $\mathfrak{t}_0$ is injective and the image contains W_K.)

Proposition 2. *We use the notation introduced above.*

(1) *We have*
$$\#\mathcal{B}_V^{\max} \leq \#\mathcal{B}_V \leq [W^\theta : W_K].$$

(2) *Assume that θ has order 2 and let $V \subset \mathfrak{g}$ be the (-1)-eigenspace, so that $\mathfrak{g} = \mathfrak{k} \oplus V$. Then*
$$\#\mathcal{B}_V^{\max} = \#\mathcal{B}_V = [W^\theta : W_K].$$

Proof. (1) follows from what has been said above. For (2) we remark that under the given assumptions we have $\mathfrak{b} = (\mathfrak{t} + \mathfrak{b}_0) \oplus V_\mathfrak{b}$ and so $\mathfrak{b}$ is is determined by $V_\mathfrak{b}$, and all $V_\mathfrak{b}$ have the same dimension, hence are maximal. □

Example 5: Quiver representations of type $\widetilde{A_1}$. Let U, W be two finite dimensional vector spaces and set $G := \mathrm{GL}(U \oplus W)$ and $\theta := \mathrm{Int}(\mathrm{id}_U, -\mathrm{id}_W)$, i.e. θ is the conjugation with $\begin{bmatrix} \mathrm{id}_U & \\ & -\mathrm{id}_W \end{bmatrix}$. Then $K = \mathrm{GL}(U) \times \mathrm{GL}(W)$ and $V = \mathrm{Hom}(U, W) \oplus \mathrm{Hom}(W, U)$ with the obvious linear action of K. By Proposition 2(2) we get $\#\mathcal{B}_V = \binom{\dim U + \dim W}{\dim U}$ and all unstable spaces in $\mathcal{B}_V$ are maximal and non-equivalent. It is known that N_V is irreducible for $m \neq n$ and has 2 irreducible components for $m = n$. A detailed analysis of the geometry of this representation can be found in [Kem82].

Remark 3. Let $\mathfrak{a} \subset V$ be a Cartan subspace, i.e., a maximal subspace consisting of semisimple elements. Then $K\mathfrak{a}$ is dense in V and the inclusion induces an isomorphism $\mathfrak{a}/\!\!/W_V \xrightarrow{\sim} V/\!\!/K^0$ where $W_V := \mathrm{Norm}_{K^0}(\mathfrak{a}) / \mathrm{Cent}_{K^0}(\mathfrak{a})$ is a finite group generated by reflections, see [Vi76]. Moreover, if $\dim \mathfrak{a} = \dim \mathfrak{t} = \mathrm{rk}\, G$, then $\mathrm{Norm}_{K^0}(\mathfrak{a})$ is finite since $\mathfrak{norm}_\mathfrak{k}(\mathfrak{a}) = \mathfrak{norm}_\mathfrak{g}(\mathfrak{a}) \cap \mathfrak{k} = \mathfrak{a} \cap \mathfrak{k} = \{0\}$, and so $\dim V = \dim K + \dim \mathfrak{a}$. It follows that in this case the generic orbits in V have finite stabilizer.

We add here a useful criterion to decide if an element $v = (v_1, v_2, \ldots, v_k) \in V^{\oplus k}$ belongs to the nullcone. A special case of this result played a fundamental rôle in the determination of the Hilbert series for the invariants of pairs of 4-qubits (see [Wa03] and the following section 4).

First of all, it is well-known and easy to see that a k-tuple $v = (v_1, v_2, \ldots, v_k)$ belongs to the nullcone of $V^{\oplus k}$ if and only if the subalgebra $\mathfrak{s} \subset \mathfrak{g}$ generated by $v_1, v_2, \ldots, v_k$ is nilpotent. (One simply uses the fact that $v = (v_1, v_2, \ldots, v_k)$ is annihilated by a 1-PSG λ if and only if the subalgebra $\mathfrak{s}$ is annihilated by λ.)

Proposition 3. *For a given $k \geq 1$ let $v = (v_1, v_2, \ldots, v_k) \in V^{\oplus k}$ and denote by $\mathfrak{s} \subset \mathfrak{g}$ the Lie algebra generated by $v_1, v_2, \ldots, v_k$. Then v belongs to the nullcone $N_{V^{\oplus k}}$ if and only if the following conditions hold:*

(1) *$v_1, v_2, \ldots, v_k \in \mathfrak{g}$ are nilpotent elements;*
(2) *The Lie algebra $\mathfrak{s} \cap \mathfrak{k}$ is nilpotent.*

Proof. One direction of the claim follows from what we just said above. Now assume that the elements $v_i \in \mathfrak{g}$ are nilpotent and that $\mathfrak{s} \cap \mathfrak{k}$ is a nilpotent Lie algebra. By construction, $\mathfrak{s}$ is θ-invariant and $\mathfrak{s} \cap \mathfrak{k} = \mathfrak{s}^\theta$. This implies, by one of the main results of [BoM55], that $\mathfrak{s}$ is solvable, hence nilpotent since it is generated by nilpotent elements. □

From now on let us assume that θ has order 2 and let V be the (-1)-eigenspace, so that $\mathfrak{g} = \mathfrak{k} \oplus V$. This is the situation studied by KOSTANT-RALLIS in [KoR71].

Remark 4. If we choose G to be the adjoint group (and θ of order 2) it is shown in [KoR71] that K has a dense orbit in the nullcone N_V. As a consequence, the number of irreducible components of N_V is less or equal to the order of the component group K/K^0. As an example let us look at the quiver representations of type $\widetilde{A_1}$ from Example 5 above. If $U = W$ and if we replace $G := \mathrm{GL}(U \oplus U)$ by its image $\bar{G}$ in $\mathrm{Aut}(\mathrm{End}(U \oplus U))$ then $\bar{K} := \bar{G}^\theta$ contains the image of the element $\begin{bmatrix} & \mathrm{id} \\ \mathrm{id} & \end{bmatrix}$ which permutes the two copies of $\mathrm{End}(U)$ in the representation of $\bar{K}$ on $V := \mathrm{End}(U) \oplus \mathrm{End}(U)$. In particular, $\bar{K}$ has a dense orbit in the (reducible) nullcone N_V, and $\bar{K}/\bar{K}^0$ has order 2.

Proposition 4. *Assume that V contains a Cartan subalgebra of $\mathfrak{g}$. Let $V_\mathfrak{b} \in \mathcal{B}_V$ be a maximal unstable subspace. Then $K^0 V_\mathfrak{b}$ is an irreducible component of N_V if and only if $V_\mathfrak{b}$ contains a regular nilpotent element of $\mathfrak{g}$. Moreover, any two different subspaces of $\mathcal{B}_V$ of this form define different irreducible components.*

Proof. We first show that V contains a regular nilpotent element n. Let $\mathfrak{a} \subset V$ be a Cartan subalgebra of $\mathfrak{g}$ and choose a Borel subalgebra $\mathfrak{b} \supset \mathfrak{a}$. Denote by $(\alpha_1, \ldots, \alpha_\ell)$ the corresponding simple roots and fix non-zero elements $e_i \in \mathfrak{g}_{\alpha_i}$. Since $\theta|_\mathfrak{a} = -\mathrm{id}_\mathfrak{a}$ we see that $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ for any root α. Choose $h \in \mathfrak{a}$ such that $[h, e_i] = 2e_i$ for all i. If $e := \sum_i a_i e_i$ and $f := \theta(e) = \sum_i a_i f_i$ where $f_i = \theta(e_i) \in \mathfrak{g}_{-\alpha_i}$ then we have $[h, f] = -2f$. Moreover, we can solve the equation $[e, f] = h$ because $[e, f] = \sum_i a_i^2 [e_i, f_i]$ and the elements $h_i := [e_i, f_i]$ form a basis of $\mathfrak{a}$. It follows that (e, f, h) is an θ-stable $\mathfrak{sl}_2$-triple. In particular, the element $n := h + e - f$ belongs to V and is nilpotent, hence conjugate to e.

As a consequence, we see that the regular nilpotent elements in N_V form a non-empty open set. Since K has a dense orbit O_K in N_V (see Remark 4 above) it follows that O_K consists of regular nilpotent elements. Hence every irreducible component contains a dense set of regular nilpotent elements. Thus $K^0\, V_\mathfrak{b}$ is an irreducible component of N_V if and only if $V_\mathfrak{b}$ contains a regular nilpotent element.

It remains to see that different subsets $V_\mathfrak{b}$ containing regular nilpotent elements define different irreducible components of N_V. So assume that $K^0 V_\mathfrak{b} = K^0 V_{\mathfrak{b}'}$ and let $x \in V_\mathfrak{b}$ be a regular nilpotent element. Then there is a $k \in K^0$ such that $x' := kx \in V_{\mathfrak{b}'}$. It follows that the regular nilpotent element x' belongs to $\mathfrak{b}'$ and also to $k\mathfrak{b}k^{-1}$. Thus $\mathfrak{b}' = k\mathfrak{b}k^{-1}$ since a regular nilpotent element belongs to a unique Borel subalgebra. It follows that $k\mathfrak{b}_0 k^{-1} = \mathfrak{b}_0$, hence $k \in B_0$ and so $\mathfrak{b} = \mathfrak{b}'$. □

Remark 5. It is easy to see that the regular nilpotent elements in V form a single K-orbit.

In addition to the assumptions of Proposition 4 assume that $\mathrm{rk}\, G = \mathrm{rk}\, K$. Then $\mathfrak{t} = \mathfrak{t}_0$ and so V is a sum of weight spaces. This implies the following combinatorial description of the irreducible components of N_V.

Corollary 1. *Assume that V contains a Cartan subalgebra of $\mathfrak{g}$ and that $\operatorname{rk} G = \operatorname{rk} K$. Then $K^0 V_{\mathfrak{b}}$ is an irreducible component of N_V if and only if $V_{\mathfrak{b}}$ contains all simple root spaces with respect to $\mathfrak{b}$.*

Example 6: Symmetric matrices. Consider the automorphism $\theta\colon A \mapsto (A^t)^{-1}$ of $G := \mathrm{SL}_n$. Then $K := G^\theta = \mathrm{O}_n$ and $V := (\mathfrak{sl}_n)_{-1}$ is the space of "traceless" symmetric $n \times n$-matrices which can be identified with $S^2(\mathbb{C}^n)/\mathbb{C}$. Clearly, V contains a Cartan subalgebra of $\mathfrak{sl}_n$, given by the diagonal matrices. Hence, we can apply Corollary 1 above and obtain the following result.

Proposition 5. *Consider the representation of SO_n on the space Sym_n of symmetric matrices. If n is even then the nullcone $N_{\mathrm{Sym}_n^{\oplus k}}$ has two irreducible components for all $k \geq 1$. They are permuted by the elements of $\mathrm{O}_n \setminus \mathrm{SO}_n$. If n is odd, then $N_{\mathrm{Sym}_n^{\oplus k}}$ is irreducible for all $k \geq 1$.*

Proof. For $n = 2\ell$, the positive roots of $\mathfrak{so}_{2\ell}$ are given by $\Delta_K^+ := \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq \ell\}$, and the weights of V by

$$X_V = \{\pm(\varepsilon_i + \varepsilon_j) \mid 1 \leq i \leq j \leq \ell\} \cup \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \leq i < j \leq \ell\} \cup \{0\}$$

where the zero weight space has dimension $\ell - 1$. The dimension of V is $\binom{n}{2} - 1 = 2\ell^2 + (\ell - 1)$. Since the representation is selfdual every maximal unstable subset $X \subset X_V$ has ℓ^2 elements. It is not difficult to see that an unstable Δ^+-invariant subset X is a subset of

$$\{\varepsilon_i \pm \varepsilon_j) \mid 1 \leq i < j \leq \ell\} \cup \{2\varepsilon_1, 2\varepsilon_2, \ldots, 2\varepsilon_\ell, -2\varepsilon_\ell\}.$$

It follows that there are exactly two non-equivalent maximal unstable Δ^+-invariant subsets,

$$\begin{aligned} X_1 &:= \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq \ell\} \cup \{2\varepsilon_1, 2\varepsilon_2, \ldots, 2\varepsilon_{\ell-1}, 2\varepsilon_\ell\} \quad \text{and} \\ X_2 &:= \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq \ell\} \cup \{2\varepsilon_1, 2\varepsilon_2, \ldots, 2\varepsilon_{\ell-1}, -2\varepsilon_\ell\}. \end{aligned}$$

It is well known that there is an element $g \in \mathrm{O}_{2\ell}$, stabilizing the given torus T_0, such that for the action on the roots we have $g\varepsilon_i = \varepsilon_i$ for $i < \ell$ and $g\varepsilon_\ell = -\varepsilon_\ell$. Hence, $gX_1 = X_2$ and so $N_{V^{\oplus k}} = \mathrm{SO}_n V_{X_1}^{\oplus k} \cup \mathrm{SO}_n V_{X_2}^{\oplus k}$ has two irreducible components for any number k of copies of V.

For $n = 2\ell + 1$ the positive roots of $\mathfrak{so}_{2\ell+1}$ are given by $\Delta_K^+ := \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq \ell\} \cup \{\varepsilon_i \mid 1 \leq i \leq \ell\}$ and the weights X_V of V by

$$\{\pm(\varepsilon_i + \varepsilon_j) \mid 1 \leq i \leq j \leq \ell\} \cup \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \leq i < j \leq \ell\} \cup \{\pm\varepsilon_i \mid 1 \leq i \leq \ell\} \cup \{0\}$$

where the zero weight space has dimension ℓ. There is exactly one maximal unstable Δ^+-invariant subset of X_V, namely

$$X := \{\varepsilon_i + \varepsilon_j \mid 1 \leq i \leq j \leq \ell\} \cup \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq \ell\} \cup \{\varepsilon_i \mid 1 \leq i \leq \ell\}.$$

Hence, the nullcone $N_{V^{\oplus k}}$ is irreducible for all $k \geq 1$. □

Example 7: Pairs of quadratic forms (the case C_m). Let U be a vector space of dimension m and consider $W := U \oplus U^*$ with the non-degenerate skew form $\beta((u,\zeta),(u',\zeta')) := \zeta'(u) - \zeta(u')$. Then $\mathrm{GL}(U)$ is naturally embedded into the corresponding symplectic group $G := \mathrm{Sp}(W,\beta)$ by $\varphi \mapsto (\varphi, (\varphi^*)^{-1})$. With $\theta := \mathrm{Int}(\mathrm{id}_U, -\mathrm{id}_{U^*})$ we find that $K := G^\theta = \mathrm{GL}(U)$ and $V := (\mathfrak{sp}(W,\beta)_{-1} = S^2(U) \oplus S^2(U^*)$. Thus $\mathrm{rk}\, G = \mathrm{rk}\, K$, and V contains a Cartan subalgebra $\mathfrak{a}$, namely

$$\mathfrak{a} := \{(\sum_i a_i u_i^2, \sum_i a_i (u_i^*)^2) \mid (a_1, a_2, \dots, a_m) \in \mathbb{C}^m\}$$

where $(u_1, \dots, u_m)$ is a basis of U and $(u_1^*, \dots, u_m^*)$ the dual basis of U^*. It follows that there are $2^m = [W_{\mathrm{Sp}(2m)} : W_{\mathrm{GL}(m)}]$ maximal unstable subsets containing a given Borel subalgebra $\mathfrak{b}_0$ of $\mathfrak{gl}(U)$.

In order to describe $\mathcal{B}_V$ we use a basis of U and its dual to obtain the usual identification $\mathfrak{sp}(W,\beta) = \mathfrak{sp}(2m) \subset \mathfrak{gl}(2m)$. Then the upper triangular matrices in $\mathfrak{sp}(2m)$ form a Borel algebra $\mathfrak{b}$. The corresponding positive roots are

$$\Delta^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le m\} \cup \{2\varepsilon_k \mid 1 \le k \le m\}$$

and contain the positive roots $\Delta_K^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le m\}$ corresponding to the Borel subalgebra $\mathfrak{b}_0 := \mathfrak{b} \cap \mathfrak{gl}(U)$. We have to describe all systems of positive roots $P \subset \Delta := \Delta^+ \cup -\Delta^+$ which contain Δ_K^+. Clearly, it suffices to describe the corresponding $\rho_P := \sum_{\alpha\in P} \alpha$, since $P = \{\alpha \in \Delta \mid (\rho_P, \alpha) > 0\}$. We have $\rho_{\Delta^+} = \sum (m-i+1)\varepsilon_i$ and $\rho_P = w\rho_{\Delta^+}$ for some $w \in W$. Hence, ρ_P has the form $\sum a_i \varepsilon_i$ where $\{|a_i|\} = \{1, 2, , \dots, m\}$.

Let us call such a P (or the corresponding ρ_P) *admissible* if it contains Δ_K^+. We have the following inductive description, starting with $\mathfrak{sp}(2) = \mathfrak{sl}(2)$ and $P = \{\varepsilon_1\}$ or $P = \{-\varepsilon_1\}$.

Proposition 6.

(1) *The admissible ρ_P's are of the form*

$$\rho_P = m\varepsilon_1 + \rho_{Q'} \qquad \text{or} \qquad \rho_P = \rho_{Q''} - m\varepsilon_m$$

where $Q' \subset \Delta' := \{\pm(\varepsilon_i - \varepsilon_j) \mid 2 \le i < j \le m\} \cup \{\pm 2\varepsilon_k \mid 2 \le k \le m\}$ and $Q'' \subset \Delta'' := \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \le i < j \le m-1\} \cup \{\pm 2\varepsilon_k \mid 1 \le k \le m-1\}$ are both admissible.

(2) *There are exactly two admissible P's such that all simple roots are in $P \cap X_V$. For $m = 2\ell$ these are given by*

$$\rho_P = m\varepsilon_1 + (m-2)\varepsilon_2 + \cdots + 2\varepsilon_{\ell-1} - \varepsilon_\ell - 3\varepsilon_{\ell+1} - \cdots - (m-1)\varepsilon_m \quad \text{or}$$
$$\rho_P = (m-1)\varepsilon_1 + (m-3)\varepsilon_2 + \cdots + \varepsilon_{\ell-1} - 2\varepsilon_\ell - 4\varepsilon_{\ell+1} - \cdots - m\varepsilon_m,$$

and for $m = 2\ell - 1$ by

$$\rho_P = m\varepsilon_1 + (m-2)\varepsilon_2 + \cdots + \varepsilon_{\ell-1} - 2\varepsilon_\ell - 4\varepsilon_{\ell+1} - \cdots - (m-1)\varepsilon_m \quad \text{or}$$
$$\rho_P = (m-1)\varepsilon_1 + (m-3)\varepsilon_2 + \cdots + 2\varepsilon_{\ell-1} - 1\varepsilon_\ell - 3\varepsilon_{\ell+1} - \cdots - m\varepsilon_m.$$

In particular, the nullcone N_V has two irreducible components.

Proof. (1) It is easy to see that the given elements are indeed of the form ρ_P for some set P of positive roots, and, by induction, we also see that they are admissible. Since their number is 2^m we found them all.

(2) We have to describe those admissible P's where no simple root of P is contained in Δ_K^+. Since $\rho_P = \sum_i a_i \varepsilon_i$ where $\{|a_i|\} = \{1, 2, \dots, m\}$ the claim easily follows. □

Again one shows that the group $\bar{K} = \bar{G}^\theta$ has two connected components, in accordance with the fact that $\bar{K}$ has a dense orbit in N_V (see Remark 5).

Example 8: Pairs of skew forms (the case D_m). Proceeding in the same way as in the previous example, this time using the non-degenerate quadratic form $q(u, \zeta) := \zeta(u)$ on $W := U \oplus U^*$ we obtain $G := \mathrm{SO}(W, q)$, $K := G^\theta = \mathrm{GL}(U)$ and $V := \bigwedge^2 U \oplus \bigwedge^2 U^*$. Again, $\operatorname{rk} G = \operatorname{rk} K$, but here V does not contain a Cartan subalgebra of $\mathfrak{so}(W, q)$. The number of admissible sets P of positive roots equals 2^{m-1}, but it is not clear which of them contribute to an irreducible component of the nullcone N_V.

§4. The Example of the 4-qubits

The group $G := \mathrm{SO}_8$ has an involution θ such that $K := G^\theta = \mathrm{SO}_4 \times \mathrm{SO}_4$ and $V = M_4$ with the obvious action of K. This representation can be identified with the 4-qubits, i.e., the representation of $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$ on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. The positive roots of $\mathfrak{k} = \mathfrak{so}_4 \oplus \mathfrak{so}_4$ are given by $\Delta_K^+ = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 + \varepsilon_2, \varepsilon_3 - \varepsilon_4, \varepsilon_3 + \varepsilon_4\}$ and the weights of V by

$$X_V = \{\pm(\varepsilon_1 \pm \varepsilon_3), \pm(\varepsilon_1 \pm \varepsilon_4), \pm(\varepsilon_2 \pm \varepsilon_3), \pm(\varepsilon_2 \pm \varepsilon_4)\}.$$

For a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{so}_8$ the half sums of positive roots $\rho_\mathfrak{b}$ has the form $\sum_i a_i \varepsilon_i$ where $\{|a_1|, |a_2|, |a_3|, |a_4|\} = \{0, 1, 2, 3\}$. Moreover, $\mathfrak{b}$ contains the Borel subalgebra $\mathfrak{b}_0 \subset \mathfrak{k}$ corresponding to Δ_K^+ if and only if $a_1 \geq \pm a_2$ and $a_3 \geq \pm a_4$. This gives the following 12 ($= [W : W_K]$) weight vectors ρ_P corresponding to the maximal unstable subspaces $V_\mathfrak{b} \in \mathcal{B}_V$:

$3\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3$	$3\varepsilon_1 + \varepsilon_2 + 2\varepsilon_3$	$3\varepsilon_1 + 2\varepsilon_3 + \varepsilon_4$
$3\varepsilon_1 - 2\varepsilon_2 + \varepsilon_3$	$3\varepsilon_1 - \varepsilon_2 + 2\varepsilon_3$	$3\varepsilon_1 + 2\varepsilon_3 - \varepsilon_4$
$\varepsilon_1 + 3\varepsilon_3 + 2\varepsilon_4$	$2\varepsilon_1 + 3\varepsilon_3 + \varepsilon_4$	$2\varepsilon_1 + \varepsilon_2 + 3\varepsilon_3$
$\varepsilon_1 + 3\varepsilon_3 - 2\varepsilon_4$	$2\varepsilon_1 + 3\varepsilon_3 - \varepsilon_4$	$2\varepsilon_1 - \varepsilon_2 + 3\varepsilon_3$

Given a $\rho_\mathfrak{b}$ from the list we know that $K^0 V_\mathfrak{b}$ is an irreducible component of N_V if and only if $V_\mathfrak{b}$ contains the simple roots, i.e. if and only if $\mathfrak{b}_0$ does not contain a simple root with respect to $\mathfrak{b}$. If ρ is of the form $\sum_i a_i\varepsilon_i$ this means that $a_1 \pm a_2 \geq 2$ and $a_3 \pm a_4 \geq 2$ which has the following four solutions:

$$O_0 := \{3\varepsilon_1 + \varepsilon_2 + 2\varepsilon_3,\ 3\varepsilon_1 - \varepsilon_2 + 2\varepsilon_3,\ 2\varepsilon_1 + 3\varepsilon_3 + \varepsilon_4,\ 2\varepsilon_1 + 3\varepsilon_3 - \varepsilon_4\}.$$

The group S_4 acts on the 4-qubits permuting the 4 factors in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. The corresponding action on $(\mathrm{SL}_2)^4$ permutes the positive (simple) roots Δ_K^+. It follows that O_0 is a single orbit under S_4 which implies that S_4 permutes the four irreducible components of N_V transitively. The remaining ρ's from the list above decompose into 2 further orbits, namely

$$O_1 := \{3\varepsilon_1 + 2\varepsilon_3 + \varepsilon_4,\ 3\varepsilon_1 + 2\varepsilon_3 - \varepsilon_4,\ 2\varepsilon_1 + \varepsilon_2 + 3\varepsilon_3,\ 2\varepsilon_1 - \varepsilon_2 + 3\varepsilon_3\} \text{ and}$$
$$O_2 := \{3\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3,\ 3\varepsilon_1 - 2\varepsilon_2 + \varepsilon_3,\ \varepsilon_1 + 3\varepsilon_3 + 2\varepsilon_4,\ \varepsilon_1 + 3\varepsilon_3 - 2\varepsilon_4\}.$$

The stabilizer of any one of the ρ's is $S_3 \subset S_4$. Since S_3 is not contained in a proper subgroup of S_4 it follows that the orbits O_i generate either none or one or four irreducible components of N_{V^k} for a given $k > 1$. In fact, we have the following result:

Proposition 7. *For $k \geq 2$ the nullcone $N_{Q_4^{\oplus k}}$ has 12 irreducible components, decomposing into 3 orbits of 4 elements under S_4. The dimensions are $8k+4$, $8k+3$ and $8k+1$.*

The proof needs a little preparation. We first have to translate the setting above into the standard coordinates of $Q_4 = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. We write $\mathbb{C}^2 = \mathbb{C}e_0 \oplus \mathbb{C}e_1$. Then the elements $e_{ijk\ell} := e_i \otimes e_j \otimes e_k \otimes e_\ell$ $(i,j,k,\ell \in \{0,1\})$ form a basis of Q_4. The corresponding weights in $K = (\mathrm{SL}_2)^4$ will be denoted by $\varepsilon_{ijk\ell}$. Note that $-\varepsilon_{ijk\ell} = \varepsilon_{i'j'k'\ell'}$ where $i + i' = j + j' = k + k' = \ell + \ell' = 1$. The non-degenerate invariant form is induced by the standard symplectic form on $\mathbb{C}^2$, hence given by

$$(e_{ijk\ell}, e_{i'j'k'\ell'}) := (i,i') \cdot (j,j') \cdot (k,k') \cdot (\ell,\ell')$$

where $(0,0) = (1,1) = 0$ and $(0,1) = -(1,0) = 1$. Such forms exist on every Q_k and are symmetric for even k and skew-symmetric for odd k. Moreover, the group S_4 acts on Q_4 by permuting the indices, hence normalizing the action of K and leaving the form $(\ ,\ )$ invariant.

The positive roots of $\mathfrak{k} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ are given by

$$\Delta_K^+ = \{\alpha_1 := \varepsilon_1 + \varepsilon_2, \alpha_2 := \varepsilon_1 - \varepsilon_2, \alpha_3 := \varepsilon_3 + \varepsilon_4, \alpha_4 := \varepsilon_3 - \varepsilon_4\},$$

and we get

$$\varepsilon_{ijk\ell} = \frac{1}{2}((-1)^i\alpha_1 + (-1)^j\alpha_2 + (-1)^k\alpha_3 + (-1)^\ell\alpha_4).$$

For $\rho_0 := 3\varepsilon_1 + \varepsilon_2 + 2\varepsilon_3 = 2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \in O_0$ the corresponding weight space $V_0 \subset Q_4$ is given by

$$V_0 = \langle e_{0000}, e_{0011}, e_{0100}, e_{0001}, e_{0010}, e_{0101}, e_{0110}, e_{1000}\rangle = (e_0 \otimes U) \oplus \mathbb{C}e_{1000}$$

where $U := \langle e_{0000}, e_{0011}, e_{0100}, e_{0001}, e_{0010}, e_{0101}, e_{0110}\rangle = (\mathbb{C}e_{0000})^{\perp}$. Similarly, we find for $\rho_1 := 3\varepsilon_1 + 2\varepsilon_3 + \varepsilon_4 \in O_1$ and $\rho_2 := 3\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3$ the following weight spaces:

$$\begin{aligned} V_1 &:= \langle e_{0000}, e_{0011}, e_{0100}, e_{0001}, e_{0010}, e_{0101}, e_{1000}, e_{1001}\rangle \\ &= \langle e_{000}, e_{001}, e_{010}, e_{100}\rangle \otimes \mathbb{C}^2, \\ V_2 &:= \langle e_{0000}, e_{0001}, e_{0010}, e_{0100}, e_{0011}, e_{0101}, e_{0110}, e_{0111}\rangle \\ &= e_0 \otimes Q_3. \end{aligned}$$

The normalizers of these spaces are $P_0 := B \times B \times B \times B$, $P_1 := B \times B \times B \times \mathrm{SL}_2$ and $P_2 := B \times \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$.

Lemma 3. *The morphisms*

$$K \times_{P_0} V_0 \to Q_4, \quad K \times_{P_1} V_1 \to Q_4, \quad K \times_{P_2} V_2 \to Q_4$$

have finite degree. In particular,

$$\dim K\, V_0^{\oplus k} = 8k + 4, \quad \dim K\, V_1^{\oplus k} = 8k + 3, \quad \dim K\, V_2^{\oplus k} = 8k + 1.$$

Proof. Let $U_- := \{\left[\begin{smallmatrix} 1 & \\ u & 1 \end{smallmatrix}\right] \mid u \in \mathbb{C}\} \subset \mathrm{SL}_2$. It suffices to check that the morphisms

$$(U_- \times U_- \times U_- \times U_-) \times V_0 \to Q_4, \quad (U_- \times U_- \times U_-) \times V_1 \to Q_4, \quad U_- \times V_2 \to Q_4$$

have rank $12 = 8 + 4$, $11 = 8 + 3$ and $9 = 8 + 1$, respectively, in a generic point of $e \in V_0, V_1$ and V_2, i.e. that

$$\begin{aligned} &\dim(\mathfrak{u}_- \oplus \mathfrak{u}_- \oplus \mathfrak{u}_- \oplus \mathfrak{u}_-) \cdot e + V_0 = 12, \\ &\dim(\mathfrak{u}_- \oplus \mathfrak{u}_- \oplus \mathfrak{u}_-) \cdot e + V_1 = 11, \\ &\dim \mathfrak{u}_- \cdot e + V_2 = 9. \end{aligned}$$

This is easy and left to the reader. □

Lemma 4. *Let $W \subsetneq Q_3$ a $B \times B \times B$-stable subspace. Then a generic 2-dimensional subspace of Q_3 is not conjugate to a subspace of W.*

Proof. $\bigwedge^2 W$ is stable under $B \times B \times B$ and $\dim \bigwedge^2 W \le \binom{7}{2} = 21$. Therefore $\dim(\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2) \bigwedge^2 W \le 21 + 3 < \binom{8}{2} = \dim \bigwedge^2 Q_3$. □

Now we are ready to give the proof of Proposition 7.

Proof of Proposition 7. According to §1 Theorem 2 (and using Lemma 4 above) we have to show the following:

(a) There is a 2-dimensional subspace of V_2 which has no conjugate in either of the spaces σV_0 or σV_1 for $\sigma \in S_4$.
(b) There is a 2-dimensional subspace of V_1 which has no conjugate in either of the spaces σV_0 for $\sigma \in S_4$.

(a) Since $V_0 = e_0 \otimes Q_3$ a conjugate of a 2-dimensional subspace of V_+0 has the form $e \otimes U$ where $e \in \mathbb{C}^2$ and U is an arbitrary 2-dimensional subspace of Q_3. It is easy to see that a subspace of σV_0 or σV_1 of this form is contained in $e \otimes W$ where W is strictly contained in Q_3. Now the claim follows from Lemma 4 above.

(b) Let $U := w \otimes \mathbb{C}^2 \subset V_1$ where w is a generic element in $\langle e_{000}, e_{001}, e_{010}, e_{100}\rangle$. Since V_0 has the form

$$V_0 = \langle e_{000}, e_{001}, e_{010}\rangle \otimes \mathbb{C}^2 \oplus (\mathbb{C}e_{0110} \oplus \mathbb{C}e_{1000})$$

it follows that a conjugate of U is contained in V_0 only if $w \in K'\langle e_{000}, e_{001}, e_{010}\rangle$ where $K' := (\mathrm{SL}_2)^3$. But $K'\langle e_{000}, e_{001}, e_{010}\rangle \subsetneq \langle e_{000}, e_{001}, e_{010,e_{100}}\rangle$ since we have $\dim K'\langle e_{000}, e_{001}, e_{010}\rangle \leq 6$, because $\langle e_{000}, e_{001}, e_{010}\rangle$ is stable under a Borel of K', and $\dim K'\langle e_{000}, e_{001}, e_{010}, e_{100}\rangle$ is an irreducible component of the nullcone N_{Q_3} of dimension 7 (see example ??).

The four images σV_0 ($\sigma \in S_4$) have the following form (we use the transpositions $\sigma = (12), (13), (14)$):

$$V_0' = \langle e_{000}, e_{001}, e_{100}\rangle \otimes \mathbb{C}^2 \oplus (\mathbb{C}e_{1010} \oplus \mathbb{C}e_{0100}),$$
$$V_0'' = \langle e_{000}, e_{100}, e_{010}\rangle \otimes \mathbb{C}^2 \oplus (\mathbb{C}e_{1100} \oplus \mathbb{C}e_{0010}),$$
$$V_0''' = \langle e_{000}\rangle \otimes \mathbb{C}^2 \oplus \langle e_{101}, e_{010}, e_{100}, e_{001}, e_{110}, e_{011}\rangle \otimes e_0,$$

and a similar argument applies. □

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Mathematisches Institut der Universität Basel
Rheinsprung 21, CH-4051 Basel, Switzerland
E-mail address: Hanspeter.Kraft@unibas.ch

Department of Mathematics, University of California, San Diego
9500 Gilman Drive, La Jolla, CA 92093-0112, USA
E-mail address: nwallach@ucsd.edu