DEGREE BOUNDS FOR INVARIANTS AND COVARIANTS OF BINARY FORMS

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Abstract. In this paper we solve an old problem of Classical Invariant Theory of binary forms. We give a modern approach and rigorous proof for the degree estimates of a generating system of invariants and covariants of binary form due to Camille Jordan. Moreover, we show that this approach can be efficiently used to calculate the generators in low degrees.

Content

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Introduction

Our base field is the field $\mathbb{C}$ of complex numbers. We denote by SL_2 the group of complex 2×2-matrices with determinant 1. All our representations of SL_2 will be complex, finite dimensional and rational. It is well known that these representations are completely reducible and that for every $n\in\mathbb{N}$ there is exactly one irreducible representation V_n of dimension $n+1$, namely the binary forms of degree n:

$$V_n = \mathbb{C}[x,y]_n = \{\text{homogeneous forms in } x \text{ and } y \text{ of degree } n\}.$$

Here $\mathbb{C}[x,y]$ is considered as the algebra of polynomial functions on $\mathbb{C}^2$ with the usual linear action of SL_2 by substitution: $(g\cdot f)(v) := f(g^{-1}v)$ for $g\in\mathrm{SL}_2,\ v\in\mathbb{C}^2$. More precisely, we find for $g=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{SL}_2$

$$g\cdot x = dx - by \qquad g\cdot y = -cx + ay.$$

0.1 Definition. Let W be a representation of SL_2. A *covariant* of W of *order* m and *degree* d is an equivariant homogeneous polynomial map $\varphi\colon W\to V_m$ of degree d, i.e., we have $\varphi(g\cdot v) = g\cdot\varphi(v)$ for all $g\in\mathrm{SL}_2$ and $\varphi(tv)=t^d\varphi(v)$ for all $t\in\mathbb{C}$.

Clearly, the covariants of order 0 are the (homogeneous) *invariant functions* on W. They form the (homogeneous components of the) *ring of invariants* which will be denoted by $\mathcal{I}(W)=\bigoplus_{d\ge 0}\mathcal{I}(W)_d$. By our definition of the representations V_n we have a natural multiplication $V_n\times V_m\to V_{n+m}$ which implies that two covariants of order n and m respectively, can be multiplied to give a covariant of order $n+m$. Thus the covariants also form a graded ring which will be denoted by $\mathcal{C}(W)=\bigoplus_{m\ge 0}\mathcal{C}(W)_m$.

The aim of this paper is to give a rigorous proof of the following important result which is due to Camille Jordan [Jor76, Jor79].

0.2 Main Theorem. *Let W be a representation of SL_2. Assume that all irreducible components of W have dimensions $\le N+1$. Then the ring $\mathcal{C}(W)$ of covariants is generated by the covariants of order $\le 2N^2$ and of degree $\le 2N^6$.*

0.3 Remark. In his famous paper [Hil93] Hilbert indicates how to obtain a general bound for the degree of a set of generators of the ring of invariants for any representation W of SL_n. Using some additional results about the structure of the invariant ring one can obtain explicit bounds in terms of data of the group and the representation (see [DeK96] for a survey). Applied to binary forms this gives a bound which depends exponentially on the dimension of W.
In a recent paper Harm Derksen has been able to improve these bounds in a fundamental way, see [Der97]. He gives a very original construction of a homogeneous system of parameters which allows to obtain polynomial bounds for the degree of a generating system.

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§1. Covariants and U-invariants

There are several other ways to look at covariants which are all equivalent and can be found in the literature. The following two are of particular interest for us.

Let $U := \{\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}\} \subset \mathrm{SL}_2$ be the maximal unipotent subgroup of upper triangular matrices and $T := \{\begin{pmatrix} s & \\ & s^{-1} \end{pmatrix} \mid s \in \mathbb{C}^*\} \subset \mathrm{SL}_2$ the maximal torus of diagonal matrices. Denote by pr_e the projection of V_e onto the first coordinate:

$$\mathrm{pr}_e \colon V_e \to \mathbb{C}, \quad f = \sum_{i=0}^{e} a_i x^{e-i} y^i \;\mapsto\; a_0.$$

This linear function pr_e is U-invariant and of *weight* (or order) e which means that for any $t = \begin{pmatrix} s & \\ & s^{-1} \end{pmatrix} \in T$ we have $\mathrm{pr}_e(t^{-1} \cdot f) = s^e \, \mathrm{pr}_e(f)$.

It is clear now that every covariant $\varphi \colon W \to V_e$ of degree d and order e determines, by composing, a U-invariant function $f_\varphi := \mathrm{pr}_e \circ \varphi$ on W which is homogeneous of degree d and of weight e. In this way we obtain a natural isomorphism of the algebra $\mathcal{C}(W)$ of covariants of W with the algebra of U-invariant functions on W, preserving degree and order. Denote by $\mathcal{O}(W)$ the algebra of polynomial functions on W, i.e. the symmetric algebra $\mathrm{S}\, W^*$ on the dual representation. Then we have in a canonical way

$$\mathcal{I}(W) = \mathcal{O}(W)^{\mathrm{SL}_2} \subset \mathcal{C}(W) = \mathcal{O}(W)^U \subset \mathcal{O}(W).$$

This interpretation of covariants has the advantage that it easily carries over to an arbitrary algebra $\mathcal{A}$ endowed with a rational and locally finite action of SL_2 by means of algebra automorphisms.

Another way is to understand a covariant of W of degree d and order e as an SL_2-stable subspace $V \subset \mathcal{O}(W)_d$ isomorphic to V_e or, more precisely, as an SL_2-equivariant linear map ${V_e}^* \to \mathcal{O}(W)_d := \{\text{homogeneous functions of degree } d\}$. With this interpretation it is not so clear how to define the product of two covariants. On the other hand, we can talk about ideals in $\mathcal{O}(W)$ generated by certain covariants which is an important concept behind Jordan's method as we will see in the next section.

We summarize our considerations above in the following proposition whose easy proof is left to the reader.

1.1 Proposition. *Every covariant $\varphi \colon W \to V_e$ (of degree d and order e) gives rise to an U-invariant function $f_\varphi := \mathrm{pr}_e \circ \varphi$ (of degree d and weight e) and to* *an equivariant linear map* $\varphi^*\colon V_e^* \to \mathcal{O}(W)_d$*. These maps* $\varphi \mapsto f_\varphi$ *and* $\varphi \mapsto \varphi^*$ *determine isomorphisms between the corresponding spaces:*

$$\mathcal{C}(W) = \mathcal{O}(W)^U = \bigoplus_{e\geq 0} \operatorname{Hom}_{\mathrm{SL}_2}(V_e, \mathcal{O}(W)).$$

§2. General setup and outline of proof

We are working with *graded* $\mathbb{C}$*-algebras* $\mathcal{R} = \bigoplus_{i\geq 0} \mathcal{R}_i$ carrying a rational and locally finite action of the group SL_2 by means of algebra automorphisms respecting the grading. (We always assume that $\dim \mathcal{R}_i < \infty$.) Such an algebra will be shortly called an SL_2*-algebra*. As mentioned before, there are two important subalgebras, the ring of invariants $\mathcal{R}^{\mathrm{SL}_2}$ and the ring of covariants (or U-invariants) $\mathcal{R}^U$, both again graded.

Let $I \subset \mathcal{R}$ be a homogeneous SL_2-equivariant ideal and denote by $\operatorname{gr}_I \mathcal{R}$ the associated graded ring $\bigoplus_{s\geq 0} I^s/I^{s+1}$. We assume that $\bigcap_s I^s = (0)$; this holds in case $I \cap \mathcal{R}_0 = (0)$. Since I is homogeneous each summand I^s/I^{s+1} has a natural graduation which turns $\operatorname{gr}_I \mathcal{R}$ into a graded ring[1]. An element $b \in \operatorname{gr}_I \mathcal{R}$ will be called *strongly homogeneous* if it is homogeneous with respect to both gradings, i.e. if it is a homogeneous element of a single I^s/I^{s+1}. Its degree $\deg b$ is the degree in the natural grading of I^s/I^{s+1}.

For an element $a \in \mathcal{R}$, $a \neq 0$, there is a unique $s \geq 0$ such that $a \in I^s \setminus I^{s+1}$. We put $\operatorname{gr} a := a + I^{s+1} \in I^s/I^{s+1} \subset \operatorname{gr}_I \mathcal{R}$. Similarly, we set $\operatorname{gr} \mathcal{S} := \{\operatorname{gr} a \mid a \in \mathcal{S}\}$ for any subset $\mathcal{S} \subset \mathcal{R}$ where we put $\operatorname{gr} 0 := 0$. Given two elements $a, b \in \mathcal{R}$ we either have $\operatorname{gr} a \cdot \operatorname{gr} b = 0$ or $\operatorname{gr} a \cdot \operatorname{gr} b = \operatorname{gr}(ab)$ by the very definition of the multiplication in $\operatorname{gr}_I \mathcal{R}$.

2.1 Lifting Lemma. (a) *Let* $\mathcal{S} \subset \mathcal{R}$ *be a set of homogeneous elements and assume that* $\operatorname{gr} \mathcal{S}$ *generates the algebra* $\operatorname{gr}_I \mathcal{R}$*. Then* $\mathcal{S}$ *generates* $\mathcal{R}$*.*

(b) *Let* $\mathcal{G} \subset \mathcal{R}$ *be a set of homogeneous elements and assume that* $\operatorname{gr} \mathcal{G}$ *linearly spans the algebra* $\operatorname{gr}_I \mathcal{R}$*. Then* $\mathcal{G}$ *linearly spans* $\mathcal{R}$*.*

(c) *Let* $\mathcal{T} \subset (\operatorname{gr}_I \mathcal{R})^U \subset \operatorname{gr}_I \mathcal{R}$ *be strongly homogeneous system of generators of the covariants. Then there is a homogeneous set* $\widetilde{\mathcal{T}} \subset \mathcal{R}^U$ *such that* $\operatorname{gr} \widetilde{\mathcal{T}} = \mathcal{T}$*, and every such* $\widetilde{\mathcal{T}}$ *generates* $\mathcal{R}^U$*.*

We will call such a subset $\widetilde{\mathcal{T}}$ *a lift of* $\mathcal{T}$.

2.2 Remark. In order to define a lift we can choose SL_2-stable homogeneous complements L_s of I^{s+1} in I^s for all s. This defines a linear isomorphism

$$\eta\colon \operatorname{gr}_I \mathcal{R} = \bigoplus_{s\geq 0} I^s/I^{s+1} \xrightarrow{\simeq} \bigoplus_{s\geq 0} L_s = \mathcal{R}$$

[1]This is *not* the usual grading of $\operatorname{gr}_I \mathcal{R}$ where I^s/I^{s+1} is the homogeneous part of degree s.

which is homogeneous of degree zero and SL_2-equivariant. Clearly, for every homogeneous subset $\mathcal{T} \subset (\mathrm{gr}_I \mathcal{R})^U$ the image $\eta(\mathcal{T})$ is a lift of $\mathcal{T}$.

Proof of the Lifting Lemma. (a) Let $r \in \mathcal{R}$ be a homogeneous element $\neq 0$ of degree d, $r \in I^s \setminus I^{s+1}$. Then we have $\mathrm{gr}\, r = p(\mathrm{gr}\, s_1, \mathrm{gr}\, s_2, \dots, \mathrm{gr}\, s_k)$ for a polynomial p and suitable $s_1, \dots, s_k \in \mathcal{S}$. Clearly, we can assume that every monomial appearing in p has degree d. Then $p(s_1, s_2, \dots, s_k)$ is homogeneous of degree d, too, and $r - p(s_1, s_2, \dots, s_k) \in I^{s+1}$. Now the claim follows because $\mathcal{R}_d \cap I^j = (0)$ for large j.

(b) The proof is similar to (a) and is left to the reader.

(c) Since the ideals I^s are all SL_2-stable the linear map $(I^s)^U \to (I^s/I^{s+1})^U$ is surjective. This implies the first claim and the second follows as in (a). □

Let us fix a natural number N and consider a representation W of SL_2 of the form

$$W = \oplus_{i \in I} V_{d_i}$$

where all numbers d_i are $\leq N$. Our goal is to find bounds for the degrees and orders of generators of $\mathcal{O}(W)^U$ which depend only on N.

The idea of JORDAN is based on studying the following ideal J in $\mathcal{O}(W)$.

2.3 Definition. Let $J \subset \mathcal{O}(W)$ be the ideal generated by all covariants of (positive) degree ≤ 3 and of order $\leq N_1 := \lfloor \frac{3}{4} N \rfloor$ where $\lfloor \alpha \rfloor$ denotes the largest integer $\leq \alpha$.

In order to find bounds for the degrees and orders of generators of $\mathcal{O}(W)^U$ it is enough, by the Lifting Lemma, to find such bounds for generators for the U-invariants $(\mathrm{gr}_J \mathcal{O}(W))^U$ of the associated graded ring

$$\mathrm{gr}_J \mathcal{O}(W) := \bigoplus_{s \geq 0} J^s/J^{s+1}.$$

Let $E \subset J$ be a minimal homogeneous and SL_2-stable subspace generating the ideal J, i.e., $E \xrightarrow{\simeq} J/\mathfrak{m}J$ where $\mathfrak{m} \subset \mathcal{O}(W)$ is the homogeneous maximal ideal. Then $\mathrm{gr}_J \mathcal{O}(W)$ is generated as a $\mathcal{O}(W)/J$-algebra by E. Denoting by W_1 the dual representation E^* we thus obtain a surjective equivariant homomorphism

$$\Phi\colon \mathcal{O}(W)/J \otimes \mathcal{O}(W_1) = \mathcal{O}(W)/J \otimes \mathrm{S}(E) \to \mathcal{O}(W)/J \oplus J/J^2 \oplus \cdots = \mathrm{gr}_J \mathcal{O}(W).$$

(Note that the left hand side is the symmetric algebra $\mathrm{S}_{\mathcal{O}(W)/J}(E) = \mathcal{O}(W)/J\,[E]$.) Moreover, $W_1 = \oplus_j V_{e_j}$ where all $e_j \leq N_1$ and so the bounds for degrees and orders of generating covariants of $\mathcal{O}(W_1)^U$ are known by induction. The first idea is to determine generating covariants for $(\mathcal{O}(W)/J)^U$ and then to extract the necessary information about generating covariants of the tensor product $(\mathcal{O}(W)/J \otimes \mathcal{O}(W_1))^U$ which can be carried over to $\mathrm{gr}_J \mathcal{O}(W)$ and then to $\mathcal{O}(W)$ using the Lifting Lemma.

2.4 Remark. By definition, the ideal J is generated in degree 1, 2 and 3 and so the space E, as a subspace of $\mathcal{O}(W)$, has a decomposition $E = E^{(1)} \oplus E^{(2)} \oplus E^{(3)}$ where $E^{(i)} := E \cap \mathcal{O}(W)_i$ is the homogeneous part of E of degree i. Thus the homomorphism $\Phi\colon \mathcal{O}(W)/J \otimes \mathcal{O}(W_1) \to \operatorname{gr}_J \mathcal{O}(W)$ *is not homogeneous*, but we have

$$\deg \Phi(F) \leq 3 \cdot \deg F \quad \text{for any } F \in \mathcal{O}(W)/J \otimes \mathcal{O}(W_1).$$

This has to be taken into account when comparing degree bounds for generators of the covariants of $\mathcal{O}(W)/J \otimes \mathcal{O}(W_1)$ and of $\operatorname{gr}_J \mathcal{O}(W)$.

In order to perform our induction we will have to iterate this procedure. We first define the numbers $N_0 := N, N_1, \ldots, N_\tau$ by $N_{i+1} = \lfloor \frac{3}{4} N_i \rfloor$ until we reach $N_\tau = 1$. Then we define inductively the graded SL_2-algebras $\mathcal{A}_i := \mathcal{O}(W_i)$ and ideals $J_i \subset \mathcal{A}_i$ as follows. Start with

$$W_0 := W, \;\; \mathcal{A}_0 := \mathcal{O}(W) \;\text{ and }\; J_0 := J.$$

Suppose that $\mathcal{A}_{i-1} = \mathcal{O}(W_{i-1})$ and J_{i-1} are defined. Then put

$$W_i := (J_{i-1}/\mathfrak{m}_{i-1} J_{i-1})^* \;\text{ and }\; \mathcal{A}_i := \mathcal{O}(W_i)$$

where $\mathfrak{m}_{i-1}$ is the homogeneous maximal ideal in $\mathcal{A}_{i-1}$, and define the ideal $J_i \subset \mathcal{A}_i$ to be generated by the covariants of (positive) degree ≤ 3 and of order $\leq N_{i+1}$. For $i = \tau$ we just put $J_\tau = 0$.

The link between the consecutive algebras $\mathcal{A}_i$ is provided by an equivariant surjective homomorphisms

$$\Phi_i : \mathcal{A}_i/J_i \otimes \mathcal{A}_{i+1} \to \operatorname{gr}_{J_i} \mathcal{A}_i := \oplus_{s \geq 0} J_i^s / J_i^{s+1}$$

which is defined, as above, by choosing an SL_2-stable homogeneous complement E_{i+1} of $\mathfrak{m}_i J_i$ in J_i, i.e. a minimal homogeneous and SL_2-stable subspace generating the ideal J_i.

Finally, we define the algebra

$$\mathcal{B} := \mathcal{A}_0/J_0 \otimes \mathcal{A}_1/J_1 \otimes \cdots \otimes \mathcal{A}_{\tau-1}/J_{\tau-1} \otimes \mathcal{A}_\tau.$$

Notice that $\mathcal{B} = \mathcal{A}_0/J_0 \otimes \mathcal{B}_1$ where $\mathcal{B}_1$ is the algebra we would obtain by the construction above if we would start our procedure with the representation W_1 instead of W.

The relation between $\mathcal{B}$ and the original algebra $\mathcal{A} = \mathcal{O}(W)$ is as follows. For arbitrary $i = 0, 1, \ldots, \tau$ consider the tensor product

$$\mathcal{D}_i := \mathcal{A}_0/J_0 \otimes \mathcal{A}_1/J_1 \otimes \cdots \otimes \mathcal{A}_{i-1}/J_{i-1} \otimes \mathcal{A}_i$$

and the ideal

$$K_i := \mathcal{A}_0/J_0 \otimes \mathcal{A}_1/J_1 \otimes \cdots \otimes \mathcal{A}_{i-1}/J_{i-1} \otimes J_i \subset \mathcal{D}_i.$$

Clearly, we have $\mathcal{D}_0 = \mathcal{A}_0 = \mathcal{O}(W)$ and $\mathcal{D}_\tau = \mathcal{B}$. Moreover, the associated graded algebra $\operatorname{gr}_{K_i} \mathcal{D}_i := \oplus_{s\geq 0} K_i^s/K_i^{s+1}$ can be identified with

$$\mathcal{A}_0/J_0 \otimes \mathcal{A}_1/J_1 \otimes \ldots \otimes \mathcal{A}_{i-1}/J_{i-1} \otimes (\oplus_{s\geq 0} J_i^s/J_i^{s+1})$$

which, in turn, is a factor of $\mathcal{D}_{i+1}$: There is a surjective SL_2-equivariant homomorphism

$$\Phi_i \colon \mathcal{D}_{i+1} \longrightarrow \operatorname{gr}_{K_i} \mathcal{D}_i$$

defined as before by choosing a homogeneous SL_2-stable complement of $\mathfrak{m}_i J_i$ in J_i. Applying the Lifting Lemma and descending induction on i we see that we have proved the following result.

2.5 Proposition. *Let $\mathcal{S} = \{C_1, C_2, \ldots\}$ be a set of homogeneous covariants generating $\mathcal{B}^U$. By consecutively taking images under $\Phi_i \colon \mathcal{D}_{i+1} \to \operatorname{gr}_{K_i} \mathcal{D}_i$, decomposing them into strongly homogeneous parts and then lifting those to covariants of $\mathcal{D}_i$ we obtain a set $\widetilde{\mathcal{S}} = \{\widetilde{C_1}, \widetilde{C_2}, \ldots\}$ of homogeneous covariants of $\mathcal{D}_0 = \mathcal{O}(W)$ which generate $\mathcal{O}(W)^U$.*

The basic result in Jordan's proof is the following description of the covariants of the algebra $\mathcal{B} = \mathcal{A}_0/J_0 \otimes \mathcal{A}_1/J_1 \otimes \cdots \otimes \mathcal{A}_{\tau-1}/J_{\tau-1} \otimes \mathcal{A}_\tau$.

2.6 RST-Theorem. *Every covariant of the algebra $\mathcal{B}$ is a linear combination of products RST where the factors R, S and T have the following form:*

- *R is a covariant of order $< 2N^2$ whose sum of order and degree is $< 9N^2$;*
- *S is a product of factors of degree 1 (and order $\leq N$) or of degree 2 and order at least 2 and at most $2N-2$;*
- *T is a product of invariants of degree $\leq 7N-5$.*

The proof of the RST-Theorem will be given in §9. It is obtained by induction using the tensor decomposition $\mathcal{B} = \mathcal{O}(W)/J \otimes \mathcal{B}_1$ (see above) together with a description of the covariants of $\mathcal{O}(W)/J$ which is given in the PQ-Theorem in §8. The proofs of both Theorems are based on the *symbolic method* which is the subject of the following sections.

Using the RST-Theorem above we can now formulate and prove the main result of Jordan's paper.

2.7 Main Theorem. *Let $N \geq 4$ and let $W = \oplus V_{d_i}$ where all $d_i \leq N$. Then every covariant of W is a linear combination of products RST where the factors R, S and T are as follows:*

- *R is a covariant of order $o < 2N^2$ and degree $d < (9N^2 - o)3^\tau$;*
- *S is a product of covariants of order $\leq 2N-2$ and of degree $\leq 2 \cdot 3^\tau$;*
- *T is a product of invariants of degree $< (7N-5)3^\tau$.*

2.8 Remark. It is not difficult to see that this results implies the bounds given in the Main Theorem 0.2 of the Introduction. In fact, $N_\tau = 1$ and $N_i \le (\frac{3}{4})^i N$ and so $\tau \le \tau_0 := \min\{i \mid (\frac{3}{4})^i N < 2\}$. This means that $(\frac{4}{3})^{\tau_0} > \frac{N}{2}$ and $(\frac{4}{3})^{\tau_0 - 1} \le \frac{N}{2}$. Hence

$$\tau \le \tau_0 = \left\lfloor \frac{\log \frac{N}{2}}{\log \frac{4}{3}} + 1 \right\rfloor \le \frac{\log_3 \frac{N}{2}}{\log_3 \frac{4}{3}} + 1 = c \cdot \log_3 N - d$$

where $c := (\log_3 \frac{4}{3})^{-1} \approx 3.81884$ and $d := c \cdot \log_3 2 - 1 \approx 1.40942$. Thus we see that the degree d of any generators listed in the Main Theorem is bounded by

$$d \le 9N^2\, 3^\tau = 3^{2-d}\, N^{2+c} \le 1.92 \cdot N^{5.82} < 2N^6.$$

Proof of the Main Theorem. Let us first prove the following "induction step":

Assume that for some i every covariant of $\mathcal{D}_{i+1}$ is a linear combination of products RST where the factors R, S and T are as follows:

- *R is a covariant of order $o < 2N^2$ and degree $d < (9N^2 - o)3^\sigma$;*
- *S is a product of covariants of order $\le 2N - 2$ and of degree $\le 2 \cdot 3^\sigma$;*
- *T is a product of invariants of degree $< (7N - 5)3^\sigma$.*

Then the same holds for $\mathcal{D}_i$ with σ replaced by $\sigma + 1$.

Start with a product RST in $\mathcal{D}_{i+1}$ as above and decompose the images $\Phi_i(R)$, $\Phi_i(S)$, $\Phi_i(T)$ in $\operatorname{gr}\mathcal{D}_i$ into strongly homogeneous components. Then each such component $\bar{R}$, $\bar{S}$ or $\bar{T}$ satisfies the corresponding condition above with σ replaced by $\sigma+1$ (see Remark 2.4: the degree of $\bar{R}, \dots$ is calculated with respect to the grading of $\mathcal{D}_i$), and the same holds for the lifts $\tilde{R} = \eta_i(\bar{R})$, $\tilde{S} = \eta_i(\bar{S})$, $\tilde{T} = \eta_i(\bar{T})$. Moreover, $\operatorname{gr}(\tilde{R}\tilde{S}\tilde{T}) = \bar{R}\bar{S}\bar{T}$, i.e., $\tilde{R}\tilde{S}\tilde{T}$ is a lift of $\bar{R}\bar{S}\bar{T}$. Since the products $\bar{R}\bar{S}\bar{T}$ linearly span the algebra $\operatorname{gr}\mathcal{D}_i$ by assumption it follows from the Lifting Lemma that the products $\tilde{R}\tilde{S}\tilde{T}$ linearly span $\mathcal{D}_i$. This proves the induction step.

Our Main Theorem now follows by descending induction on i. We start with $i = \tau$ where $\mathcal{D}_\tau = \mathcal{B}$ and the claim is true by the RST-Theorem 2.6, and end with $i = 0$ where $\mathcal{D}_0 = \mathcal{O}(W)$ and the claim is our Main Theorem. □

2.9 Remark. By a more detailed analysis of the first steps in the induction procedure one can show that in the Main Theorem 2.7 the exponent τ can be replaced by ρ which is defined to be the first integer such that $N_{\rho-1} \le 4$. It is easy to see that ρ is either $\tau - 1$ or $\tau - 2$ depending on $N_{\rho-1} = 3$ or 4.

This slightly improves the degree bound d for the generators R, S and T in the Main Theorem: $d < 0.407 \cdot N^{5.82} < N^6$.

§3. Fundamental Theorems for SL_2 and Symbolic Expressions

The classical symbolic method which we will describe in the next section is the basic and fundamental tool to manipulate invariants and covariants of binary forms. We will use it to produce normal forms for covariants of low degree which will finally enable us to calculate degree bounds for the generating system. For forms of low degrees the symbolic method can also be used to determine an explicit minimal system of generators for the ring of covariants.

Denote by $L := V_1$ the space of linear forms and let $W = L^{\mathcal{J}} = \bigoplus_{s\in\mathcal{J}} L$ be the direct sum of copies of L parametrized by the index set $\mathcal{J}$. For an element $(l_s)_{s\in\mathcal{J}}$ we write $l_s = s_0x + s_1y$. The invariants and covariants of $L^{\mathcal{J}}$ are well-known. For a proof of the following result we refer to the literature (see [GY03] and [We46] for a rigorous approach).

3.1 First Fundamental Theorem for SL_2. *The invariants of $L^{\mathcal{J}}$ are minimally generated by the determinants*

$$(l_i)_{i\in\mathcal{J}} \mapsto [l_a\, l_b] := \det\begin{pmatrix} a_0 & b_0 \\ a_1 & b_1 \end{pmatrix}, \quad a, b \in \mathcal{J}.$$

The covariants, as an algebra over the invariants, are minimally generated by the linear projections

$$(l_i)_{i\in\mathcal{J}} \mapsto l_c, \quad c \in \mathcal{J}.$$

Like in the classical literature we denote the determinant shortly by (ab) and the linear projection by c_x:

$$(ab)\colon (l_i)_{i\in\mathcal{J}} \mapsto [l_a\, l_b] \qquad c_x\colon (l_i)_{i\in\mathcal{J}} \mapsto l_c.$$

With this notation we obtain the following corollary:

3.2 Corollary. *The covariants of $L^{\mathcal{J}}$ are linearly spanned by the maps*

$$P_{\lambda,\sigma} = \prod_{a\neq b}(ab)^{\lambda_{ab}} \prod_c c_x^{\sigma_c}, \qquad \lambda_{ab}, \sigma_c \in \mathbb{N}.$$

The order of $P_{\lambda,\sigma}$ is equal to $\sum_c \sigma_c$ and the multidegree $(p_a)_{a\in\mathcal{J}}$ is given by $p_a = \sum_b(\lambda_{ab} + \lambda_{ba}) + \sigma_a$.

3.3 Definitions. An expression of the form $P = P_{\lambda,\sigma}$ as above, i.e., a (commutative) monomial in the elements (ab) and c_x, will be called a *symbolic expression* (in the alphabet $\mathcal{J}$). The *order* $\operatorname{ord} P$ and the *weight* $\operatorname{wt} P = (\operatorname{wt}_a P)_{a\in\mathcal{J}}$ of the symbolic expression P are defined by

$$\operatorname{ord} P := \sum_c \sigma_c \quad \text{and} \quad \operatorname{wt}_a P := \sum_b(\lambda_{ab} + \lambda_{ba}) + \sigma_a.$$

In particular, $\mathrm{wt}_a P$ is equal to the number of times the symbol a occurs in the symbolic expression P. The *category* of the symbolic expression P is the maximal exponent of a determinant factor of P:

$$\operatorname{cat} P := \max_{a,b} \lambda_{a,b}.$$

The symbolic expression P is called *decomposable* if it can be written as a product $P = P_1 P_2$ in a non-trivial way where P_1 and P_2 are disjoint, i.e., no symbol occurs in both. Finally, we denote by $\operatorname{supp} P$ the *support* of P, i.e., the set of symbols $a \in \mathcal{J}$ occurring in P.

With this notation the First Fundamental Theorem says that we have a canonical surjective homomorphism

$$\mathbb{C}[(ab), c_x \mid a, b, c \in \mathcal{J}] \twoheadrightarrow \mathcal{C}(L^{\mathcal{J}}).$$

The next step is to describe the kernel of this map.

The covariants (ab) and a_x satisfy the following fundamental relations:

$$(ab) + (ba) = 0, \tag{i}$$
$$(ab)\, c_x + (bc)\, a_x + (ca)\, b_x = 0, \tag{ii}$$
$$(ab)\,(cd) + (bc)\,(ad) + (ca)\,(bd) = 0. \tag{iii}$$

The first relation is clear from the definition. Both expressions on the lefthand side of (2) and (3) are alternating in the symbols a, b, c (and d) and therefore vanish because $\dim L = 2$.

It is known that these relations form a generating system for all relations. This is the content of the following result.

3.4 Second Fundamental Theorem for SL_2**.** *There is a canonical isomorphism of (multi-) graded algebras*

$$\mathrm{Symb}_{\mathcal{J}} := \mathbb{C}[(ab), c_x \mid a, b, c \in \mathcal{J}]/\mathfrak{a} \xrightarrow{\sim} \mathcal{C}(L^{\mathcal{J}})$$

where the ideal $\mathfrak{a}$ *is generated by the elements*

$$\begin{array}{ll} (ab) + (ba) & a, b \in \mathcal{J} \\ (ab)\, c_x + (bc)\, a_x + (ca)\, b_x & a, b, c \in \mathcal{J} \\ (ab)\,(cd) + (bc)\,(ad) + (ca)\,(bd) & a, b, c, d \in \mathcal{J} \end{array}$$

(Again we refer to [GY03] and [W46] for the proof.)

The algebra $\mathrm{Symb}_{\mathcal{J}}$ defined above is called *the symbolic algebra over the alphabet* $\mathcal{J}$. It carries a canonical linear action of the symmetric group $S_{\mathcal{J}}$ (by permuting the symbols), and the map is equivariant (with respect to the obvious action on $\mathcal{C}(L^{\mathcal{J}})$.)

The Second Fundamental Theorem allows an interesting interpretation of the space spanned by the symbolic expressions P of a fixed order e and a fixed weight $(d_1, d_2, \dots, d_n)$ where $n = |\mathcal{J}|$ is the number of symbols. Putting $F = \mathbb{C}^n$ we have $L^{\mathcal{J}} = F \otimes L$ and $\mathcal{C}(L^{\mathcal{J}}) = \mathcal{O}(F \otimes L)^U$. From CAUCHY's formula (see [GW98, Cor. 4.5.19]) we get the decomposition

$$\mathcal{O}(F \otimes L)_d = \bigoplus_{\substack{\nu=(2^u,1^v)\\ 2u+v=d}} S_\nu(F^*) \otimes S_\nu(L^*)$$

where S_ν denotes the SCHUR functor associated to the partition ν. (We shortly write $(2^u, 1^v)$ for the partition $(\underbrace{2, 2, \dots, 2}_{u \text{ times}}, \underbrace{1, 1, \dots, 1}_{v \text{ times}})$.) Since $S_{(2^u,1^v)}(L^*) = V_v$, we see that the covariants of degree d and order e can be identified with the space $S_\nu(F^*)$ for $\nu = (2^f, 1^e)$, $f := \frac{1}{2}(d - e)$. Using the natural action of the diagonal torus of $\mathrm{GL}(F) = \mathrm{GL}_n$ we finally get the following result.

3.5 Corollary. *There is a natural identification between the space spanned by the symbolic expressions P of order e and weight $\delta = (d_1, d_2, \dots, d_n)$ and the corresponding weight space in the* SCHUR *functor $S_\nu(\mathbb{C}^n)$ where $\nu = (2^f, 1^e)$ and $f = \frac{1}{2}(\sum_i d_i - e)$.*

As a consequence, all identities involving symbolic expressions can be understood as identities in certain weight spaces of SCHUR functors and can be checked for example by using their expressions in some standard basis (see [Bu98]).

Let us mention here that this is a special case of the *Howe duality*. However, the above identification was already known to the classics (cf. [GY03, chap. II]).

§4. SYMBOLIC METHOD

We will show that symbolic expressions can be used in a very efficient way to describe and manipulate invariants and covariants of binary forms.

Fix a decomposition $W = \oplus_{i \in I} V_{d_i}$ into irreducible components. Every covariant of W is a sum of *multihomogeneous* covariants. Moreover, a multihomogeneous covariant φ of multidegree (m_i) can be polarized to produce a *multilinear* covariant $\tilde{\varphi}$ of the representation $\tilde{W} = \oplus_{i \in I} V_{d_i}^{m_i}$ where each V_{d_i} is repeated m_i many times. Let us denote this space by $\oplus_{a \in \mathcal{J}} V_{p_a}$. Clearly, φ can be reconstructed from $\tilde{\varphi}$ by "diagonally" embedding W into $\tilde{W}$:

$$\varphi(\dots, v_i, \dots) = \tilde{\varphi}(\dots, \underbrace{v_i, v_i, \dots, v_i}_{m_i \text{ times}}, \dots)$$

4.1 Remark. Polarizing a covariant does not change its order nor its degree. Therefore, it would be sufficient to consider only *multilinear covariants*, i.e., covariants which are of degree 1 or 0 in each variable, and to show that they can be expressed as polynomials in those covariants satisfying the bound conditions for degree and order formulated in the Main Theorem 2.7. However, it will be more convenient to consider the ring of all covariants of W and to use its multiplicative structure.

As before let $L := V_1$ be the space of linear forms on $\mathbb{C}^2$. For every d we have the power map $L \to V_d$ given by $l \mapsto l^d$. They determine an equivariant morphism

$$\pi\colon L^{\mathcal{J}} \to \tilde{W} = \oplus_{s\in\mathcal{J}} V_{p_s}, \quad (l_s)_{s\in\mathcal{J}} \mapsto (l_s^{p_s})_{s\in\mathcal{J}}.$$

Since the representation V_{p_s} can be identified with the symmetric power $S^{p_s}L$ we see that the multilinear functions on $\tilde{W}$ correspond to the multihomogeneous functions on $L^{\mathcal{J}}$. Thus every multilinear covariant $\tilde{\varphi}$ of $\tilde{W}$ is completely determined by its pull-back $\Phi := \pi^*(\tilde{\varphi}) = \tilde{\varphi}\circ\pi$ which is a covariant of $L^{\mathcal{J}}$ of the same order and of weight $(p_a)_{a\in\mathcal{J}}$. Thus, every multihomogeneous covariant φ of multidegree (m_i) is completely determined by Φ.

Conversely, every covariant of $L^{\mathcal{J}}$ of multidegree $(p_a)_{a\in\mathcal{J}}$ and order e determines a multilinear covariant of $\tilde{W}$ of the same order and therefore a multihomogeneous covariant of W of multidegree (m_i) and order e.

In the previous section we gave a description of the covariants of $L^{\mathcal{J}}$. It follows from the First Fundamental Theorem 3.1 that Φ is a linear combination of covariants of the form

$$P_{\lambda,\sigma} = \prod_{a\neq b} (ab)^{\lambda_{ab}} \prod_c c_x^{\sigma_c}$$

where $\lambda_{ab}, \sigma_c \in \mathbb{N}$ are subject to the conditions

$$\operatorname{wt}_a P = \sum_b (\lambda_{ab} + \lambda_{ba}) + \sigma_a = p_a \quad \text{for } a \in \mathcal{J} \quad \text{and} \quad \operatorname{ord} P = \sum_c \sigma_c = e.$$

The multidegree $(m_i)_{i\in I}$ of the corresponding covariant φ is recovered from the fact that in the index set $\mathcal{J}$ there are exactly m_i symbols a corresponding to $i \in I$. Thus we see that the total degree of φ is given by the number of symbols which occur in the symbolic expression P. We call this the *degree* of the symbolic expression P:

$$\deg P := \text{number of different symbols occuring in } P = \#\operatorname{supp} P$$

Thus, we finally obtain the following result which summarizes what is classically called "symbolic method":

4.2 Proposition. *Consider a symbolic expression*

$$P = P_{\lambda,\sigma} = \prod_{a \neq b} (ab)^{\lambda_{ab}} \prod_c c_x^{\sigma_c}$$

of weight $(p_a)_{a \in \mathcal{J}}$*, order* e *and degree* m*. Associate to every symbol* a *occurring in* P *an element* $i \in I$ *such that* $d_i = p_a$ *and denote by* m_i *the number of elements associated* i*. Then* P *determines a multihomogeneous covariant* φ_P *of* $W = \sum_i V_{d_i}$ *of multidegree* m_i *and order* e*. Conversely, every covariant of* W *is a linear combination of such* φ_P *associated to symbolic expressions* P*.*

It will always be clear in the context which index $i \in I$ has to be associated to a symbol $a \in \operatorname{supp} P$.

If $\varphi = \varphi_P$ we say that P is the *symbolic expression of the covariant* φ, although the expression P is in general not uniquely determined by φ because of the fundamental relations. It might even happened that φ_P is zero.

4.3 Remark. If the covariant φ has symbolic expression $P = \prod_{a \neq b} (ab)^{\lambda_{ab}} \prod_c c_x^{\sigma_c}$ then its value on $(f_a)_{a \in \mathcal{J}} \in \tilde{W} = \oplus_{a \in \mathcal{J}} V_{p_a}$ is calculated in the following way:

> *Write* $f_a = l_{a_1} l_{a_2} \cdots l_{a_{p_a}}$ *as a product of linear forms and replace in the symbolic expression* P *the* p_a *symbols* a *by the different symbols* $a_1, a_2, \ldots, a_{p_a}$ *in all* $p_a!$ *possible ways. Then replace the brackets* $(a_i b_j)$ *by the determinants* $[l_{a_i}\, l_{b_j}]$*, the symbols* $a_{i\,x}$ *by the linear forms* l_{a_i}*, sum over all these forms and divide by* $\prod_a p_a!$ *(which is the number of summands):*

$$\varphi((f_a)_{a \in \mathcal{J}}) = \frac{1}{\prod_a p_a!} \sum_{i,j,k} \left(\prod_{a \neq b} [l_{a_i}\, l_{b_j}]^{\lambda_{ab}} \prod_a l_{a_k}^{\sigma_{a_k}} \right).$$

In order to calculate φ on $(f_i)_{i \in I} \in W$ we proceed in the same way, but identify $f_a = f_b = f_c = \cdots := f_i$ for all symbols $a, b, c, \ldots$ which are associated to the same index $i \in I$, and use a fixed decomposition $f_i = l_1 l_2 \cdots l_{m_i}$ for all of them.
(It is clear that this is the correct description of the map φ: It is well defined because it is symmetric in the l_{a_i}, it is equivariant by construction, and it coincides with φ on the forms $(f_a = l_a^{p_a})_{a \in \mathcal{J}}$, by definition.)

It is best to look now at some examples.

4.4 Examples.
(1) The linear projection $W = \sum_i V_{d_i} \to V_{d_k}$ has symbolic expression $a_x^{d_k}$. (The symbol a is associated to $k \in I$.)
(2) The *Jacobian*

$$(f, h) \mapsto \operatorname{Jac}(f, h) := \det \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{pmatrix}$$

is a bilinear covariant of $V_p \oplus V_q$ of order $p+q-2$. It has symbolic expression $(ab)a_x^{p-1}b_x^{q-1}$, up to the factor pq. This is easily seen by using Remark 4.3 above:

$$\begin{aligned}\mathrm{Jac}(l_1l_2\cdots l_p, m_1m_2\cdots m_q) &= \sum_{i,j}[l_i, m_j]\, l_1\cdots\widehat{l_i}\cdots l_p\, m_1\cdots\widehat{m_j}\cdots m_q \\ &= \frac{1}{(p-1)!(q-1)!}\sum [l_{i_1}, m_{j_1}]\, l_{i_2}\cdots l_{i_p} m_{j_2}\cdots m_{j_q} \\ &= pq\cdot\varphi_{(ab)a_x^{p-1}b_x^{q-1}}(l,m).\end{aligned}$$

(3) Let $q = \alpha_0x^2 + 2\alpha_1xy + \alpha_2y^2 \in V_2$ be a quadratic form. The *discriminant* $\Delta(q) := \alpha_0\alpha_2 - \alpha_1^2$ is an invariant; it has symbolic expression $2(ab)^2$.
In fact, write $q = l_al_b$. Then, by Remark 4.3,

$$\begin{aligned}\varphi_{(ab)^2}(q) &= \frac{1}{4}([l_a\, l_a][l_b\, l_b] + [l_a\, l_b][l_b\, l_a] + [l_b\, l_a][l_a\, l_b] + [l_b\, l_b][l_a\, l_a]) \\ &= -\frac{1}{2}[l_a\, l_b]^2 = -\frac{1}{2}(a_0b_1 - a_1b_0)^2 \\ &= \frac{1}{2}(4a_0b_0a_1b_1 - (a_0b_1 + a_1b_0)^2) = 2(\alpha_0\alpha_2 - \alpha_1^2).\end{aligned}$$

A more direct argument is the following: The covariant φ with symbolic expression $(ab)^2$ does not vanish on the forms $q = l_1l_2$ with linearly independent l_1, l_2, but it vanishes on the forms $q = l^2$. Thus φ is a multiple of the discriminant.

Similarly, one shows that the symbolic expression $(ab)^2(bc)(cd)^2(da)$ determines a non-zero invariant φ of V_3 of degree 4 which vanishes on all forms $f \in V_3$ having a linear factor of multiplicity ≥ 2. Thus φ is a multiple of the discriminant.

On the other hand the symbolic expression $P = (ab)(ac)(ad)(bc)(bd)(cd)$ represents the zero function on V_3: Permuting the symbols a and b gives $-P$, but represents the same invariant on V_3.

(4) The symbolic expressions $(ab)^4$ and $(ab)^2(bc)^2(ca)^2$ determine two non-zero invariants φ and ψ of V_4 of degree 2 and 3, respectively. For $f \in V_4$, $f = \alpha_0x^4 + 4\alpha_1x^3y + 6\alpha_2x^2y^2 + 4\alpha_3xy^3 + \alpha_4y^4$ they are, up to a factor, explicitly given by

$$\varphi(f) = \alpha_0\alpha_4 - 4\alpha_1\alpha_3 + 3\alpha_2^2$$

and

$$\psi(f) = \det\begin{pmatrix}\alpha_0 & \alpha_1 & \alpha_2\\ \alpha_1 & \alpha_2 & \alpha_3\\ \alpha_2 & \alpha_3 & \alpha_4\end{pmatrix}$$

and are classically called *“Apolare”* and *“Hankelsche Determinante”*.

(5) The *Hessian*

$$f \mapsto \mathrm{Hess}(f) := \det\begin{pmatrix}\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x\partial y}\\ \frac{\partial^2 f}{\partial x\partial y} & \frac{\partial^2 f}{\partial y^2}\end{pmatrix}$$

is a covariant of V_n of degree 2 and order $2n-4$ which has symbolic expression $(ab)^2 a_x^{n-2} b_x^{n-2}$, up to a factor. (Again this follows easily from Remark 4.3.)
(6) Let $V = V_p \oplus V_q$, $p \geq q$. Then the symbolic expressions

$$(ab)^k \, a_x^{p-k} b_x^{q-k} \quad k = 0, 1, \dots, q$$

determine bilinear covariants

$$\varphi^{(k)}: V \to V_{p+q-2k}.$$

These are obviously all possible symbolic expressions giving bilinear covariants of $V_p \oplus V_q$, and they are all non-zero. Thus we obtain an explicit form of the well-known formula of Clebsch-Gordan

$$V_p \otimes V_q \simeq V_{p+q} \oplus V_{p+q-2} \oplus \cdots \oplus V_{p-q}$$

describing the SL_2-equivariant decomposition of the tensor product $V_p \otimes V_q$.
(7) Let $V = V_{p_1} \oplus V_{p_2} \oplus \cdots \oplus V_{p_m}$ be a representation of SL_2 and define

$$\mathcal{N}_V := \{(\alpha_1 \ell^{p_1}, \alpha_2 \ell^{p_2}, \cdots, \alpha_m \ell^{p_m}) \mid \alpha_i \in \mathbb{C}, \ell \in L\}.$$

It is easy to see that every *invariant* symbolic expression $P = \prod_{a \neq b} (ab)^{\lambda_{ab}}$ vanishes on $\mathcal{N}_V$ (use Remark 4.3). Moreover, one shows that the converse is true, too: *$\mathcal{N}_V \subset V$ is the zero set of the homogeneous invariants of positive degree.* This subset $\mathcal{N}_V$ is usually called the *nullcone* of V.

4.5 Remark. The symbolic method, as explained above, applies to invariants and covariants of representations W of SL_2. However, every SL_2-algebra $\mathcal{R}$ can be presented in the form $\mathcal{R} = \mathcal{O}(W)/J$ where $W \subset \mathcal{R}$ is a suitable SL_2-representation generating $\mathcal{R}$, and thus the symbolic method can be used here as well. Of course, it depends on the choice of W. Moreover, we see that every generating system for the covariants of W gives a generating system for the covariants of $\mathcal{R}$.

§5. Transvections

The Clebsch-Gordan decomposition which we described explicitly in the previous section (§4 Example 6) gives some ways to combine covariants. Consider the following equivariant bilinear map (we assume $p \geq q$):

$$[\ ,\ ]_k : \ V_p \times V_q \longrightarrow V_p \otimes V_q \xrightarrow{\simeq} V_{p+q} \oplus V_{p+q-2} \oplus \cdots \oplus V_{p-q} \xrightarrow{\mathrm{pr}} V_{p+q-2k}$$

which is defined for $0 \leq k \leq q = \min\{p, q\}$. If φ and ψ are covariants of V of order p and q respectively, then the map

$$[\varphi, \psi]_k : V \xrightarrow{(\varphi,\psi)} V_p \times V_q \xrightarrow{[\ ,\ ]_k} V_{p+q-2k}$$

is a covariant of order $p+q-2k$. This covariant is classically called the *kth transvection* of φ and ψ (German: *"Überschiebung"*). Clearly, the 0th transvection is just multiplication: $[\varphi,\psi]_0 = \varphi\psi\colon V \to V_{p+q}$.

Given two covariants with symbolic expression P and Q we want to give an explicit description of their transvections in terms of the symbolic expressions. For this we can assume that P and Q are disjoint, i.e. they have no symbol in common. For example, if $\varphi\colon V \to V_{p_a}$ and $\psi\colon V \to V_{p_b}$ are linear projections, i.e., φ has symbolic expression a_x^p and ψ symbolic expression b_x^q, then the transvection $[\varphi,\psi]_k$ is a bilinear covariant of order $p+q-2k$ and has symbolic expression $(ab)^k\, a_x^{p-k} b_x^{q-k}$:

$$[\varphi_{a_x^p}, \varphi_{b_x^q}]_k = \varphi_{(ab)^k\, a_x^{p-k} b_x^{q-k}}.$$

(This is just our definition, see §4 Example 6.) We shortly write for this

$$[a_x^p, b_x^q]_k = (ab)^k\, a_x^{p-k} b_x^{q-k}.$$

Thus we see that this transvection is obtained from the product $a_x^p b_x^q$ by replacing k products of the form $a_x b_x$ by the determinant factor (ab). This is a special case of the following construction.

5.1 Definition. A *k-fold contraction* (shortly: *k-contraction*) of two (disjoint) symbolic expressions P and Q is a symbolic expression T obtained from the product PQ by choosing k pairs (a_x, b_x) where a_x occurs in P and b_x occurs in Q and replacing their product $a_x b_x$ by the determinant factor (ab).

A *k-contraction* of a single symbolic expression P is a symbolic expression P' obtained from P by choosing k products $a_x b_x$ in P where $a \neq b$ and replacing them by the corresponding determinant factor (ab).

Clearly, the 0-contraction of P and Q is just the product PQ.

5.2 Example. If $P = a_x b_x c_x$ and $Q = e_x f_x$ are disjoint then the 1-contractions are

$$(ae)b_x c_x f_x,\ (af)b_x c_x e_x,\ (be)a_x c_x f_x,\ (bf)a_x c_x e_x,\ (ce)a_x b_x f_x,\ (cf)a_x b_x e_x.$$

It is easy to show that the first transvection $[\varphi_P, \varphi_Q]_1$ of the corresponding covariants φ_P and φ_Q has the symbolic expression

$$\frac{1}{6}((ae)b_x c_x f_x + (af)b_x c_x e_x + (be)a_x c_x f_x + (bf)a_x c_x e_x + (ce)a_x b_x f_x + (cf)a_x b_x e_x).$$

In general, we have the following result which is an immediate consequence of our Remark 4.3 in §4 and Definition 5.1 of the transvection .

5.3 Lemma A. *Let P and Q be two disjoint symbolic expressions. The transvection $[\varphi_P, \varphi_Q]_k$ is a linear combination of covariants φ_T where T runs through the k-fold contractions of P and Q, and each such φ_T occurs with a positive rational coefficient q_T where $\sum q_T = 1$:*

$$[\varphi_P, \varphi_Q]_k = \sum_T q_T \varphi_T, \quad q_T \in \mathbb{Q}_{>0} \text{ and } \sum_T q_T = 1.$$

From now on we will often confuse P and φ_P and use P instead of φ_P in order to simplify the notation. Thus the above formula reads

$$[P, Q]_k = \sum_T q_T\, T.$$

The next claim is an easy consequence of the fundamental relation (i)-(iii) in 3.3.

5.4 Lemma B. *Let T and T' be two k-fold contractions of two disjoint symbolic expressions P and Q. Then the difference $T - T'$ can be written as an integral linear combination of lower contractions. More precisely,*

$$T - T' = \sum_j T_j$$

and each T_j is either a $(k-1)$-fold contraction of P_j and Q_j where P_j is P and Q_j a 1-contraction of Q or Q_j is Q and P_j a 1-contraction of P, or T_j is a $(k-2)$-fold contraction of P_j and Q_j where both are 1-contractions of P and Q, respectively.

Proof. By definition T and T' are both obtained from the product PQ by choosing k pairs (a_x, b_x) where a_x occurs in P and b_x occurs in Q and replacing their product $a_x b_x$ by the determinant factor (ab). It is easy to see that we can find k-contractions $T_0 = T, T_1, T_2, \dots, T_n = T'$ such that each pair (T_i, T_{i+1}) is of one of the following forms:

(1) $T_i = S(ab)a'_x$ and $T_{i+1} = S(a'b)a_x$ where $P = P_0 a_x a'_x$, $Q = Q_0 b_x$ and S is a $(k-1)$-contraction of $P_0 Q_0$. It follows from relation (ii) in 3.3 that $T_i - T_{i+1} = S(aa')b_x$. This last symbolic expression is clearly a $(k-1)$-contraction of $P' = P_0(aa')$ with $Q = Q_0 b_x$.

(2) $T_i = S(ab)(a'b')$ and $T_{i+1} = S(ab')(a'b)$ where $P = P_0 a_x a'_x$, $Q = Q_0 b_x b'_x$ and S is a $(k-2)$-contraction of $P_0 Q_0$. In this case we use relation (iii) in 3.3 to get $T_i - T_{i+1} = S(aa')(bb')$. This symbolic expression is a $(k-2)$-contraction of $P' = P_0(aa')$ with $Q' = Q_0(bb')$. □

5.5 Lemma C. *Let T be an k-fold contraction of two (disjoint) symbolic expressions A and B. Then*

$$T - [A, B]_k = \sum_j r_j T_j = \sum_\ell p_\ell [A_\ell, B_\ell]_{k_\ell}$$

where T_j is obtainded from A and B as in Lemma B above, $k_\ell < k$, $p_\ell \in \mathbb{Q}_{\geq 0}$, and A_ℓ and B_ℓ are contractions of A and B, respectively.

Proof. The claim is obvious for $k = 0$, so we can use induction. By Lemma A we have $[A, B]_k = \sum_i q_i S_i$ where $q_i \in \mathbb{Q}_{>0}$ and $\sum_i q_i = 1$. Thus

$$T - [A, B]_k = \sum_i q_i(T - S_i).$$

Now Lemma B implies that $T - S_i$ is a sum of $(k-1)$ or $(k-2)$-contractions T_j of A_j and B_j which are contractions of A and B, respectively. This gives the first equation. For the second, we find by induction $T_j = [A_j, B_j]_{k'} + \sum_{k_\ell < k'} p_\ell [A_\ell, B_\ell]_{k_\ell}$, $(k' = k - 1$ or $k - 2)$ which has the required form. □

The next lemma is an immediate consequence of the previous result. We just state it for completeness.

5.6 Lemma D. *Let T and T' be two k-fold contractions of two (disjoint) symbolic expressions A and B. Then*

$$T - T' = \sum_j r_j [A_j, B_j]_{k_j}$$

where $k_j < k$, $r_j \in \mathbb{Q}$, and A_j and B_j are contractions of A and B, respectively.

As an application we give a "constructive" proof of the following result which plays a central role in Gordan's proof of the finiteness of invariants and covariants of binary forms. We will also use in an essential way in the last section (§10) where we give the examples.

The proof of the proposition is based on Gordan's famous Lemma. (Recall that a symbolic expression C is *decomposable* if it can be written as a product of two disjoint symbolic expressions, see 3.3.)

5.7 Proposition. *Let $\mathcal{R}$ and $\mathcal{S}$ be two SL_2-algebras whose covariants are finitely generated. Then the covariants of the tensor product $\mathcal{R} \otimes \mathcal{S}$ are finitely generated, too. If $P_1, P_2, \ldots, P_r$ are generators of the covariants of $\mathcal{R}$ and $Q_1, Q_2, \ldots, Q_s$ generators of the covariants of $\mathcal{S}$, then a finite generating system can be chosen from the set of transvections $[P, Q]_\ell$ $(\ell \geq 0)$ where P is a monomial in the P_i's and Q a monomial in the Q_j's.*

More precisely, assume that the generators P_i, Q_j are given as symbolic expressions. Define a set $\mathcal{G}$ of symbolic expressions in the following way: For every pair P, Q of monomials and every $\ell \leq \operatorname{ord} P, \operatorname{ord} Q$ choose an ℓ-contraction of P and Q, but only in case there does not exist a decomposable ℓ-contraction of P and Q. Then this set $\mathcal{G}$ is finite and is a generating system for the covariants of $\mathcal{R} \otimes \mathcal{S}$.

Proof. It is clear from Definition 5.1 of the transvection that the covariants of $\mathcal{R}\otimes\mathcal{S}$ are linearly spanned by the transvections $[\varphi, \psi]_\ell$ where φ is a covariant of $\mathcal{R}$ and ψ

a covariants of $\mathcal{S}$, hence also from the transvections $[P,Q]_\ell$ where P runs through the monomials in the P_i's and Q through the monomials in the Q_j's. Thus the first claim follows from the more precise statement in the second part.

Now assume that the P_i, Q_j are given in symbolic form (cf. Remark 4.5; we also assume that $\mathcal{R}$ and $\mathcal{S}$ are graded). For every triple P, Q, ℓ where $\ell \leq \operatorname{ord} P, \operatorname{ord} Q$ we choose an ℓ-contraction $C_\ell(P,Q)$. Then the set $\{C_\ell(P,Q)\}$ linearly spans $(\mathcal{R}\otimes\mathcal{S})^U$. In fact, let $\mathcal{T}_k \subset (\mathcal{R}\otimes\mathcal{S})^U$ be the subspace spanned by all $[P,Q]_\ell$ and $\mathcal{C}_k \subset (\mathcal{R}\otimes\mathcal{S})^U$ the subspace spanned by all $C_\ell(P,Q)$ where $\ell \leq k$ and P and Q are monomials in the P_i and Q_j, respectively. Then $\mathcal{T}_k$ contains all transvections $[A,B]_j$ where $j \leq k$, A is a covariant of $\mathcal{R}$ and B a covariant of $\mathcal{S}$. Moreover, $(\mathcal{R}\otimes\mathcal{S})^U = \bigcup \mathcal{T}_k$, and $\mathcal{C}_0 = \mathcal{T}_0$ is the subalgebra generated by $\{P_1,\dots,P_r,Q_1,\dots,Q_s\}$, because $[P,Q]_0 = PQ = C_0(P,Q)$. We will show that $\mathcal{C}_\ell \supset \mathcal{T}_\ell$ for all ℓ. By induction, we can assume that $\mathcal{C}_{\ell-1} \supset \mathcal{T}_{\ell-1}$. Given $[P,Q]_\ell \in \mathcal{T}_\ell$ we have $C_\ell(P,Q) - [P,Q]_\ell \in \mathcal{T}_{\ell-1} \subset \mathcal{C}_{\ell-1}$ by Lemma C and so $[P,Q]_\ell \in \mathcal{C}_\ell$ which proves the claim.

Now let $\mathcal{G} \subset \{C_\ell(P,Q)\}$ be the subset defined by removing all $C_\ell(P,Q)$ with the property that there exists a decomposable ℓ-contraction of P and Q. By induction on the degree one easily sees that $\mathcal{G}$ generates the algebra of covariants.

In order to finish the proof we have to show that $\mathcal{G}$ is finite, i.e., that for almost all triples P,Q,ℓ there is an ℓ-contraction of P and Q which is decomposable. Let $P = P_{i_1}P_{i_2}\cdots P_{i_a}$ and $Q = Q_{j_1}Q_{j_2}\cdots Q_{j_b}$ and assume that we can write $\ell = \ell' + \ell''$ and find decompositions $\{1,2,\dots,a\} = R' \cup R''$, $\{1,2,\dots,b\} = S' \cup S''$ in such a way that

$$\sum_{\nu\in R'} \operatorname{ord} P_{i_\nu},\ \sum_{\mu\in S'} \operatorname{ord} Q_{i_\mu} \geq \ell' \quad \text{and} \quad \sum_{\nu\in R''} \operatorname{ord} P_{i_\nu},\ \sum_{\mu\in S''} \operatorname{ord} Q_{i_\mu} \geq \ell''.$$

Then we can choose an ℓ'-contraction C' of $P' := \prod_{\nu\in R'} P_{i_\nu}$ and $Q' := \prod_{\mu\in S'} Q_{j_\mu}$ and an ℓ''-contraction C'' of $P'' := \prod_{\nu\in R''} P_{i_\nu}$ and $Q'' := \prod_{\mu\in S''} Q_{j_\mu}$ and thus obtain a decomposable ℓ-contraction $C = C'C''$ of P and Q.

Now put $d_i := \operatorname{ord} P_i$ and $e_j := \operatorname{ord} Q_j$. The preceding considerations show that if $P = P_1^{m_1}\cdots P_r^{m_r}$ and $Q = Q_1^{n_1}\cdots Q_s^{n_s}$ then there exists an ℓ-contraction of P and Q if and only if we have

$$\sum_i m_i d_i = \ell + k_1 \quad \text{and} \quad \sum_j n_j e_j = \ell + k_2 \tag{$*$}$$

where $k_1, k_2 \geq 0$, and there exists a decomposable ℓ-contraction of P and Q if and only if we can find positive decompositions $m_i = m_i' + m_i''$, $n_j = n_j' + n_j''$, $\ell = \ell' + \ell''$, $k_1 = k_1' + k_1''$, $k_2 = k_2' + k_2''$ such that

$$\sum_i m_i' d_i = \ell' + k_1', \qquad \sum_j n_j' e_j = \ell' + k_2',$$
$$\sum_i m_i'' d_i = \ell'' + k_1'', \qquad \sum_j n_j'' e_j = \ell'' + k_2''.$$

By GORDAN's famous Lemma the equations $(*)$ have only finitely many "indecomposable" positive solutions in m_i, n_j, ℓ, k_1, k_2. Hence there are only finitely many triples (P, Q, ℓ) such that there exists an indecomposable ℓ-contraction of P and Q. This finally finishes the proof of the proposition. □

5.8 Remark. We can also say something without assuming finite generation. In fact, the proof of the previous proposition gives the following: Assume that the covariants of $\mathcal{R}$ are generated by those of degree $\leq d_{\mathcal{R}}$ and order $\leq e_{\mathcal{R}}$ and similarly for $\mathcal{S}$. Then there are numbers d and e which can be calculated from $d_{\mathcal{R}}, e_{\mathcal{R}}, d_{\mathcal{S}}, e_{\mathcal{S}}$ such that the covariants of $\mathcal{R} \otimes \mathcal{S}$ are generated by those of degree $\leq d$ and order $\leq e$.

§6. Covariants of degree 3

We start with an easy lemma. Consider the graded algebra

$$R := \mathbb{C}[x, y, z]/(x + y + z) = \oplus_{n \geq 0} R_n$$

where R_n denotes the nth homogeneous component of R.

6.1 Lemma. *Let $n + 1 = n_1 + n_2 + n_3$ where $0 \leq n_1, n_2, n_3 \leq n$. Then the monomials*

$$\{x^i y^{n-i} \mid i < n_1\} \cup \{y^j z^{n-j} \mid j < n_2\} \cup \{z^k x^{n-k} \mid k < n_3\}$$

form a basis of R_n.

Proof. Clearly, the dimension of R_n is $n + 1 = n_1 + n_2 + n_3$ which is equal to the number of monomials listed in the lemma. Replacing z by $-(x + y)$ and then y by 1 it suffices to show that the polynomials

$$\mathcal{P}_{n_1,n_2,n_3} := \{x^i \mid i < n_1\} \cup \{(x + 1)^{n-j} \mid j < n_2\} \cup \{(x + 1)^k x^{n-k} \mid k < n_3\}$$

are either linearly independent in $\mathbb{C}[x]$ or span the subspace of all polynomials of degree $\leq n$. It is easy to see that the derivatives of these functions span the same space as the set of polynomials

$$\mathcal{P}_{n_1-1,n_2,n_3} = \{x^i \mid i < n_1-1\} \cup \{(x+1)^{n-1-j} \mid j < n_2\} \cup \{(x+1)^k x^{n-1-k} \mid k < n_3\}.$$

(We can assume that $n_1 > 0$.) Since the constant 1 belongs to $\mathcal{P}_{n_1,n_2,n_3}$ the claim follows by induction on n. □

The lemma will be applied to manipulate symbolic expressions in the following way. Consider the expressions

$$P_{\alpha,\beta,\gamma} := (ab)^\gamma (bc)^\alpha (ca)^\beta \, a_x^{p_a-\beta-\gamma} b_x^{p_b-\alpha-\gamma} c_x^{p_c-\alpha-\beta}.$$

If $p_a, p_b, p_c \geq n := \alpha + \beta + \gamma$ then we can write

$$P_{\alpha,\beta,\gamma} = ((ab)c_x)^\gamma \cdot ((ca)b_x)^\beta \cdot ((bc)a_x)^\alpha \cdot a_x^{p_a-n} b_x^{p_b-n} c_x^{p_c-n}.$$

The fundamental relation (ii) in 3.3 shows that there is a well-defined homomorphism from R_n to the linear span of the $P_{\alpha,\beta,\gamma}$ by sending $x^\gamma y^\alpha z^\beta$ to $P_{\alpha,\beta,\gamma}$. Therefore, the lemma above implies that every such P is a linear combination of symbolic expressions P' with only two determinant factors. Hence, we get the following result.

6.2 Proposition. *Assume that $p_a, p_b, p_c \geq n := \alpha+\beta+\gamma$ and let $n+1 = n_1+n_2+n_3$ where the n_i are non-negative integers. Then every symbolic expression*

$$(ab)^\gamma (bc)^\alpha (ca)^\beta \, a_x^{p_a-\beta-\gamma} b_x^{p_b-\alpha-\gamma} c_x^{p_c-\alpha-\beta}$$

of degree 3 is a rational linear combination of the following three types of symbolic expressions:

(i) $(ab)^{n-i}(bc)^i \, a_x^{p_a-n+i} b_x^{p_b-n} c_x^{p_c-i}$ *where* $i < n_1$*;*
(ii) $(bc)^{n-j}(ca)^j \, a_x^{p_a-j} b_x^{p_b-n+j} c_x^{p_c-n}$ *where* $j < n_2$*;*
(iii) $(ca)^{n-k}(ab)^k \, a_x^{p_a-n} b_x^{p_b-k} c_x^{p_c-n+k}$ *where* $k < n_3$*.*

(If $n_s = 0$ then the corresponding term is not needed.)

In particular, choosing $n_1, n_2, n_3 \leq \frac{n}{3}+1$—this is always possible!—we see that all symbolic expressions occurring in (i), (ii) and (iii) have category $\geq \frac{2n}{3}$. (Recall that the category of a symbolic expression Q is the maximal exponent of a determinant factor occurring in Q, see Definition 3.3.)

6.3 Corollary. *If $p_a, p_b, p_c \geq n := \alpha + \beta + \gamma$ then the symbolic expression $P = (ab)^\gamma (bc)^\alpha (ca)^\beta \, a_x^{p_a-\beta-\gamma} b_x^{p_b-\alpha-\gamma} c_x^{p_c-\alpha-\beta}$ is a rational linear combination of symbolic expressions of category $\geq \frac{2}{3}n$ with only two determinantal factors. Moreover, we can assume that all terms in this linear combination have category $\geq \operatorname{cat} P$.*

Proof. It remains to prove the last statement. It is clear if $\operatorname{cat} P \leq \frac{2}{3}n$. Otherwise let us assume that $\operatorname{cat} P = \alpha > \frac{2}{3}n$. The fundamental relation says that $P_{\alpha,\beta,\gamma} + P_{\alpha+1,\beta,\gamma-1} + P_{\alpha,\beta+1,\gamma-1} = 0$. Thus $P_{\alpha,\beta,\gamma}$ is equal to $\pm P_{\alpha,\beta+\gamma,0}$ modulo terms of category $> \alpha$. Now the claim follows by induction since the statement is clear for $\operatorname{cat} P = n$. □

Before giving the main result of this section let us introduce the following two ideals $I \subset J$ of the coordinate ring $\mathcal{O}(W)$ where $W = \oplus_i V_{d_i}$, $d_i \leq N$. As before, we denote by N_1 the integral part of $\frac{3}{4}N$. The first ideal I is generated by all covariants of degree ≤ 2 and order $\leq N_1$ and the second ideal J by all covariants of degree ≤ 3 and order $\leq N_1$. Two covariants or symbolic expressions are called *equivalent modulo I or J* if their difference belongs to I or J, respectively.

6.4 Definition. Let P be a symbolic expression and let a, b be two symbols appearing in P. Then the determinant factor $(ab)^\gamma$ of P is called the *ab-factor of* P (or simply a 2-factor of P) and the integer $p_a + p_b - 2\gamma$ is called the *order* of the ab-factor. Similarly, we define the abc-factor (a 3-factor) of P to be the part $(ab)^\gamma(bc)^\alpha(ca)^\beta$ of P and its order to be the integer $p_a + p_b + p_c - 2(\alpha+\beta+\gamma)$.

6.5 Lemma. *If a symbolic expression P contains a term a_x with $p_a \le N_1$ or an ab-factor $(ab)^\mu$ of order $p_a + p_b - 2\mu \le N_1$ then P belongs to I. If it contains an abc-factor $(ab)^\gamma(bc)^\alpha(ca)^\beta$ of order $p_a+p_b+p_c-2(\alpha+\beta+\gamma) \le N_1$ then P belongs to J.*

Proof. Let us give the proof of the second claim; the others follow in a similar way. By definition, the ideal I contains all transvections $[K^\mu_{a,b}, B]_\nu$ where $K^\mu_{a,b} := (ab)^\mu a_x^{p_a-\mu} b_x^{p_b-\mu}$ has order $p_a + p_b - 2\mu \le N_1$ and B is arbitrary. By assumption, P is a contraction of such a $K^\mu_{a,b}$ with some B. Hence it follows from Lemma 5.5.C that P can be expressed as a linear combination $\sum p_i [K^{\mu_i}_{a,b}, B_i]_{\nu_i}$ where $\mu_i \ge \mu$ and $\nu_i \le \nu$. Thus all $K^{\mu_i}_{a,b}$ have order $\le N_1$ and the claim follows. □

6.6 Example. If the symbolic expression P has category $\ge \frac{5}{8}N$ then P belongs to I. (In fact, if $(ab)^\mu$ is a 2-factor where $\mu \ge \frac{5}{8}N$ then the order of this ab-factor is $p_a + p_b - 2\mu \le 2(N - \frac{5}{8}N) = \frac{3}{4}N$.)

Similarly, if P contains an abc-factor $(ab)^\gamma(bc)^\alpha(ca)^\beta$ such that $\alpha+\beta+\gamma \ge \frac{9}{8}N$ then P belongs to J.

Now we come to the main result of this section. We call it the abc-Theorem.

6.7 abc-Theorem. *Let $P = (ab)^\gamma(bc)^\alpha(ca)^\beta\, a_x^{p_a-\beta-\gamma} b_x^{p_b-\alpha-\gamma} c_x^{p_c-\alpha-\beta}$ be a symbolic expression of degree 3. Assume that $p_a \ge p_b \ge p_c$ and put $n := \alpha+\beta+\gamma = \frac{1}{2}(p_a+p_b+p_c - \operatorname{ord} P)$.*

(a) *If $n \le p_c$ then P is a linear combination of symbolic expressions of the form $(ef)^\mu(fg)^\nu e_x^{d_e-\mu} f_x^{d_f-\mu-\nu} g_x^{d_g-\nu}$ where $\mu+\nu = n$, $\mu \ge 2\nu$ and $\mu \ge \operatorname{cat} P$.*

(b) *If $p_c \le n \le p_b$ then P is modulo I equivalent to a linear combination of expressions of the form $(ab)^\mu(bc)^\nu a_x^{p_a-\mu} b_x^{p_b-\mu-\nu} c_x^{p_c-\nu}$ where $\mu+\nu = n$, $\mu \ge 2\nu$ and $\mu \ge \operatorname{cat} P$.*

(c) *If $p_b \le n \le p_a$ then P belongs to I.*

(d) *If $p_a \le n$ then P belongs to J.*

Proof. We first remark that the condition $\mu \ge 2\nu$ in (a) and (b) is equivalent to the condition $\mu \ge \frac{2}{3}n$ (and to $\nu \le \frac{1}{3}n$) because $\mu+\nu = n$.

(a) This is Corollary 6.3 above.

(b) Set $p_c = n - \lambda$. Then $\lambda \le n - p_c \le (\alpha+\beta+\gamma) - (\alpha+\beta) = \gamma$. Thus we can write

$$P = (ab)^\lambda a_x^\lambda b_x^\lambda\, P'$$

where $P' = P'_{\alpha',\beta',\gamma'}$ is defined as $P_{\alpha,\beta,\gamma}$, using $p'_a := p_a - \lambda, p'_b := p_b - \lambda, p'_c := p_c$ where $\alpha' := \alpha, \beta' := \beta, \gamma' := \gamma - \lambda$. In particular, we have $n' := \alpha' + \beta' + \gamma' = n - \lambda = p_c \leq p'_a, p'_b, p'_c$. Hence, we can apply Corollary 6.3 to P' and find, after multiplication with $(ab)^\lambda a_x^\lambda b_x^\lambda$, that P is a linear combination of symbolic expressions which contain one of the following factors

$$(ab)^{\mu+\lambda}(bc)^\nu \quad (bc)^\mu(ca)^\nu \quad (ca)^\mu(ab)^{\nu+\lambda}$$

where $\mu \geq \frac{2}{3}n'$. In the first case we are done: $\mu + \nu = n'$ and so $\mu \geq 2\nu$. In the second and third case we claim that the corresponding symbolic expression belongs to I. In fact, the bc-factor in case 2 and the ca-factor in case 3 have order $\leq N_1$:

$$\begin{aligned} p_b + p_c - 2\mu \leq p_a + p_c - 2\mu \leq p_a + p_c - \frac{4}{3}n' = p_a + p_c - \frac{4}{3}p_c \\ = p_a - \frac{1}{3}p_c < N - \frac{1}{3}N_1 \leq N_1 + 1. \end{aligned}$$

(We used here that $p_c > N_1$ since otherwise $P \in I$. Moreover, the last inequality follows from $4N_1 \leq 3N \leq 4N_1 + 3$ which implies $N - \frac{1}{3}N_1 \leq (\frac{4}{3}N_1 - 1) - \frac{1}{3}N_1 = N_1 + 1$.)

(c) This case and the next are similar to (b) except that the calculations are more involved. Let $p_c = n - \lambda$ and $p_b = n - \rho$. As before, we find $\lambda \leq \gamma$ and $\rho \leq \beta$ and so P can be written in the form $P = (ab)^\lambda (ca)^\rho\, a_x^{\lambda+\rho} b_x^\lambda c_x^\rho\, P'$ where $P' = P'_{\alpha',\beta',\gamma'}$ with

$$\begin{aligned} p'_a &= p_a - \lambda - \rho, & p'_b &= p_b - \lambda, & p'_c &= p_c - \rho, \\ \alpha' &= \alpha - \lambda - \rho, & \beta' &= \beta - \lambda, & \gamma' &= \gamma - \rho. \end{aligned}$$

Hence, we have $n' := \alpha' + \beta' + \gamma' = n - \lambda - \rho = p'_c = p'_b \leq p'_a$. In particular, $n' = (n - \lambda) + (n - \rho) - n \geq 2(N_1 + 1) - N \geq \frac{1}{2}(N + 1)$ and so $n' \geq 2$ since we can assume that $N \geq 2$. Now we choose non-negative integers m_1, m_2, m_3 such that $m_1 + m_2 + m_3 = n' - 2$. Then it follows from Proposition 6.2 above that P is a linear combination of symbolic expressions which contain one of the following factors

$$(ab)^{n'-m_1+\lambda+\rho} \qquad (bc)^{n'-m_2+\lambda} \qquad (ca)^{n'-m_3+\rho}.$$

The claim will follow if we show that we can choose m_1, m_2, m_3 in such a way that the corresponding 2-factors have order $\leq N_1$, i.e., that the following inequalities are satisfied:

$$\begin{aligned} p_a + p_b - 2(n' - m_1 + \lambda) &\leq N_1 \\ p_b + p_c - 2(n' - m_2) &\leq N_1 \\ p_c + p_a - 2(n' - m_3 + \rho) &\leq N_1 \end{aligned} \tag{$*$}$$

This will follow from the following two claims:

(1) *The three inequalities* $(*)$ *are satisfied if we replace each* m_i *by zero.*
(2) *The sum of the left hand sides of the inequalities* $(*)$ *is less or equal to* $3N_1 - 3$.

(We need $3N_1-3$ in (2) because the parity of N_1 could be different from the parity of all sums p_a+p_b, p_b+p_c and p_c+p_a and then the inequalities $(*)$ are all strict.)
(1) First we have $p_b+p_c-2n'=0$. Moreover,

$$p_a+p_b-2(n'+\lambda)=p_a+p_b-2(n-\rho)=p_a-p_b\le N-N_1\le N_1$$

and similarly $p_c+p_a-2(n'+\rho)\le N_1$. (Again we assume $p_b,p_c>N_1$ since otherwise $P\in I$.)
(2) The sum s of the left hand sides of $(*)$ is

$$\begin{aligned}s&=2(p_a+p_b+p_c)-6n'+2(m_1+m_2+m_3)-2(\lambda+\rho)\\&=2(p_a+p_b+p_c)-4n'-2(\lambda+\rho)-4\\&=2(p_a+p_b+p_c)-2(n'+\lambda)-2(n'+\rho)-4\\&=2p_a-4\le 2N-4\le 3N_1-3.\end{aligned}$$

This settles the case (c).

(d) As in the previous cases put $p_c=n-\lambda, p_b=n-\rho, p_a=n-\tau$ and define $n':=n-\lambda-\rho-\tau=p_c+\rho+\tau=p_b+\lambda+\tau=p_a+\lambda+\rho$. We can assume that the order of P is $>N_1$ since otherwise P belongs to J and we are done. Hence, $N_1<p_a+p_b+p_c-2n=p_a+p_b+p_c-2(n'+\lambda+\rho+\tau)=n'$ and so $n'\ge 2$. With the same arguments as above we see that the claim follows if we can find non-negative integers m_1,m_2,m_3 such that $m_1+m_2+m_3=n'-2$ and that

$$\begin{aligned}p_a+p_b-2(n'-m_1+\lambda)&\le N_1\\p_b+p_c-2(n'-m_2+\tau)&\le N_1\\p_c+p_a-2(n'-m_3+\rho)&\le N_1\end{aligned}\qquad(**)$$

Again, we first have to show that the inequalities are satisfied if we put $m_i=0$:

$$\begin{aligned}p_a+p_b-2(n'+\lambda)&=p_a+p_b-(n-\rho)-(n-\tau)+\rho+\tau\\&=\rho+\tau=(n-\lambda)-n'\le N-N_1\le N_1\end{aligned}$$

(We used again that $n'>N_1$.) Now let s be the sum of the left hand sides of $(**)$:

$$\begin{aligned}s&=2(p_a+p_b+p_c)-6n'+2(m_1+m_2+m_3)-2(\lambda+\tau+\rho)\\&=2(p_a+p_b+p_c)-4n'-2(\lambda+\tau+\rho)-4\\&=2(p_a+p_b+p_c)-(n'+\lambda+\rho)-(n'+\lambda+\tau)-(n'+\rho+\tau)-n'-4\\&=p_a+p_b+p_c-n'-4\\&\le 3N-N_1-5\le 3N_1-2.\end{aligned}$$

Thus we get $s\le 3N_1-3$ except possibly in the case where $p_a=p_b=p_c=N$, $n'=N_1+1$ and $s=3N_1-2$. But then we have equality in one of the equations $(**)$ and so N_1 is even. In this case, the inequality $s\le 3N_1$ is sufficient to find a solution of $(**)$ with the required properties, as we remarked above. This completes the proof of the theorem. □

We want to apply the above theorem to general symbolic expressions.

6.8 Proposition. *Let* $(ab)^\gamma(bc)^\alpha(ca)^\beta$ *be a* 3*-factor of a symbolic expression* P *of degree* ≥ 3 *and put* $n := \alpha+\beta+\gamma$*. Then* P *is equivalent modulo* J *to a linear combination of symbolic expressions* P' *whose abc-factor has the form* $(ef)^\mu(fg)^\nu$ *where* (e,f,g) *is a permutation of* (a,b,c)*,* $\mu \geq 2\nu$*,* $\mu \geq \max(\alpha,\beta,\gamma)$ *and* $p_e, p_f \geq \mu+\nu \geq n$*. In particular,* $\operatorname{cat} P' \geq \mu \geq \frac{2}{3}n$*.*

Proof. For the given 3-factor of $(ab)^\gamma(bc)^\alpha(ca)^\beta$ of P we can assume that $p_a \geq p_b \geq p_c$. Thus, P is a ρ-contraction of $P_{\alpha,\beta,\gamma}$ with some symbolic expression Q. If $\rho = 0$ then the claims follow immediately from the *abc*-Theorem applied to $P_{\alpha,\beta,\gamma}$. So, by induction, we can assume that the proposition holds for every ρ'-contractions of any $P_{\alpha',\beta',\gamma'}$ with an arbitrary symbolic expression Q' where $\rho' < \rho$.

We now claim that the following assertions are equivalent:

(i) *The proposition holds for one* ρ*-contraction of* $P_{\alpha,\beta,\gamma}$ *with* Q*.*
(ii) *The proposition holds for all* ρ*-contractions of* $P_{\alpha,\beta,\gamma}$ *with* Q*.*
(iii) *The transvection* $[P_{\alpha,\beta,\gamma}, Q]_\rho$ *is a linear combination of symbolic expressions* P' *which satisfy the conditions of the proposition.*

Clearly, (ii) implies (i) and, by Lemma 5.3.A, also (iii). Moreover, by Lemma 5.5.C, we get for an arbitrary ρ-contraction P of $P_{\alpha,\beta,\gamma}$ with Q

$$P - [P_{\alpha,\beta,\gamma}, Q]_\rho = \sum_i p_i [P_{\alpha_i,\beta_i,\gamma_i}, Q_i]_{\rho_i} \qquad (**)$$

where $\alpha_i \geq \alpha, \beta_i \geq \beta, \gamma_i \geq \gamma$ and $\rho_i < \rho$. Thus, again by Lemma 5.3.A, the right hand side is a linear combination of ρ'-contractions P' of certain $P_{\alpha',\beta',\gamma'}$ with some Q' where $\rho' < \rho$, $\alpha'+\beta'+\gamma' \geq \alpha+\beta+\gamma = n$ and $\operatorname{cat} P' \geq \max(\alpha',\beta',\gamma') \geq \max(\alpha,\beta,\gamma)$. By induction, the proposition applies to the right hand side of the equation $(**)$ and so (i) and (iii) are equivalent. But P was an arbitrary contraction and thus (ii) follows, too.

Now we can finish the proof of the proposition. We use the *abc*-Theorem to write $P_{\alpha,\beta,\gamma} = \varphi_0 + \sum q_i P_i$ where P_i is of the form $(ef)^\mu(fg)^\nu e_x^{p_e-\mu} f_x^{p_f-\mu-\nu} g_x^{p_g-\nu}$ ($\mu \geq 2\nu$, $\mu \geq \max(\alpha,\beta,\gamma)$ and $p_e, p_f \geq \mu+\nu = n$) and φ_0 belongs to J. Then any contraction of P_i with Q has the required form and so claim (iii) holds for the transvections $[P_i, Q]_\rho$ by Lemma 5.3.A. Thus, it holds for the transvection $[P_{\alpha,\beta,\gamma}, Q]_\rho$ by linearity and the fact that $[\varphi_0, Q]_\rho$ belongs to J. Now the implication (iii) $\Rightarrow$ (ii) above shows that the proposition holds for P. □

The previous proposition above implies the following result.

6.9 Corollary. *Every covariant of degree* ≥ 3 *is equivalent modulo* J *to a linear combination of symbolic expressions* P *which satisfy* $\operatorname{cat} P \geq \frac{2}{3}\tau_3(P)$ *where* $\tau_3(P)$ *is defined to be the maximum of the sum of the exponents* $\alpha+\beta+\gamma$ *of an arbitrary 3-factor* $(ab)^\gamma(bc)^\alpha(ca)^\beta$ *of* P*.*

(In fact, choose the *abc*-term $(ab)^\gamma(bc)^\alpha(ca)^\beta$ in P such that $\alpha+\beta+\gamma = \tau_3(P)$. Then P is a linear combination of symbolic expressions P' of the form given in

Proposition 6.8, and so $\operatorname{cat} P' \geq \mu \geq 2\nu$ and $\mu + \nu \geq \tau_3(P)$. But this implies that $\operatorname{cat} P' \geq \frac{2}{3}\tau_3(P)$.)

6.10 Remark. A similar kind of induction as in the proof above will be used again in the following two sections.

§7. Covariants of degree 4

The aim of this section is to describe some properties of 4-factors in symbolic expressions. We start with the following *normal form* for covariants of degree 4 modulo the ideal J.

7.1 Proposition. *Every covariant φ of degree 4 is modulo the ideal J equivalent to a linear combination of symbolic expressions of the form*

$$(ab)^\mu (bc)^\nu (ad)^\sigma \; a_x^{p_a-\mu-\sigma} b_x^{p_b-\mu-\nu} c_x^{p_c-\nu} d_x^{p_d-\sigma}$$

where $\mu \geq 2\nu, 2\sigma$.

Proof. It suffices to prove the theorem for a symbolic expression P of degree 4. If Q is an arbitrary symbolic expression we define $\tau_3(Q)$ to be the maximum of $\alpha+\beta+\gamma$ where $(ab)^\gamma(bc)^\alpha(ca)^\beta$ is a 3-factor of Q (see Corollary 6.9). If $\tau_3(Q)$ is large enough then Q belongs to J (see Example 1 of §6). So we can assume that the proposition holds for all symbolic expressions P' of degree 4 with $\tau_3(P') > \tau_3(P)$.

Let $(ab)^\gamma(bc)^\alpha(ca)^\beta$ be a 3-factor of P such that $\tau_3(P) = \alpha + \beta + \gamma =: n$. Then P is a ρ-contraction of $P_{\alpha,\beta,\gamma}$ with $d_x^{p_d}$. If $\rho = 0$ then the claim follows from the *abc*-Theorem 6.7. Hence, we can assume that the proposition holds for all ρ'-contractions of an arbitrary $P_{\alpha',\beta',\gamma'}$ with $d_x^{p_d}$ where $\rho' < \rho$. Using Lemma A and C of §5 we conclude as in the proof of Proposition 6.8 of the previous section (§6) that the following claims are equivalent:

(i) *The proposition holds for one ρ-contraction of $P_{\alpha,\beta,\gamma}$ with $d_x^{p_d}$.*
(ii) *The proposition holds for all ρ-contractions of $P_{\alpha,\beta,\gamma}$ with $d_x^{p_d}$.*
(iii) *The proposition holds for the transvection $[P_{\alpha,\beta,\gamma}, d_x^{p_d}]_\rho$.*

Using the *abc*-Theorem 6.7, the equivalence (i)⇔(iii) and the linearity of the transvection reduces the proof to the case of an arbitrary ρ-contraction P of the symbolic expression $(ab)^\mu(bc)^\nu a_x^{p_a-\mu} b_x^{p_b-\mu-\nu} c_x^{p_c-\nu}$ with $d_x^{p_d}$ where $\mu \geq 2\nu$ and $p_a > n = \mu+\nu$. In particular, we have $p_a - \mu > \nu$.

Now we choose the ρ-contraction P in such a way that the exponent σ of the *ad*-factor is maximal, i.e. $\sigma = \min(p_a - \mu, \rho)$. If $\sigma > \nu$ then $\tau_3(P) \geq \mu + \sigma > \mu + \nu = n$ and the claim follows by induction. If $\sigma \leq \nu$ then P has the given form. □

A consequence of the theorem above is the following: Every covariant P of degree 4 is equivalent modulo J to a linear combination of symbolic expressions P_i where $\operatorname{cat} P_i$ is greater or equal to half of the sum of the exponents of the determinant factors of P_i, for all i. The next proposition shows that this holds in general.

7.2 Proposition. *Let $(ab)^\gamma(bc)^\alpha(ac)^\beta(ad)^\delta(bd)^\epsilon(cd)^\eta$ be a 4-factor of a symbolic expression A of degree ≥ 4 and put $m := \alpha+\beta+\gamma+\delta+\epsilon+\eta$, the sum of the exponents. Then A is equivalent modulo J to a linear combination of symbolic expressions of category $\geq \frac{m}{2}$.*

Proof. The proof is similar to the proof of Proposition 6.8 in §6. We only give the main lines.

(1) By assumption, A is a ρ-contraction of a symbolic expression P of degree 4 (corresponding to the $abcd$-factor of A) and a symbolic expression Q. For $\rho = 0$ the claim follows from the theorem above. Hence, we can assume that the proposition holds for any ρ'-contractions of an arbitrary expression P' of degree 4 with some expression Q' provided that $\rho' < \rho$.

(2) Using again the Lemmas A and C of §5 in combination with (1) we find that the following statements are equivalent:

(i) *The proposition holds for one ρ-contraction of P with Q.*
(ii) *The proposition holds for all ρ-contractions of P with Q.*
(iii) *The proposition holds for the transvection $[P,Q]_\rho$.*

Using Proposition 7.1 we can therefore assume that P is of the special form

$$(ab)^\mu(bc)^\nu(ad)^\sigma\, a_x^{p_a-\nu-\sigma} b_x^{p_b-\mu-\nu} c_x^{p_c-\nu} d_x^{p_d-\sigma}$$

where $\mu+\nu+\sigma \geq m$ and $\mu \geq 2\nu, 2\sigma$. Hence, $2\operatorname{cat} A \geq 2\mu \geq \mu+\nu+\sigma \geq m$. □

7.3 Corollary. *Every covariant φ of degree ≥ 4 is equivalent modulo J to a linear combination of symbolic expressions A which satisfy $2\operatorname{cat} A \geq \tau_4(A)$ where $\tau_4(A)$ denotes the maximum of the sum of exponents of a 4-factor of A .*

In fact, if the condition $2\operatorname{cat} A \geq \tau_4(A)$ is not satisfied for a symbolic expression A then Proposition 7.2 above shows that A is equivalent modulo J to a linear combination of symbolic expression A_i of strictly higher category.

§8. Normal form of covariants modulo J

The following result plays a crucial rôle in the proof of our Main Theorem 2.7. It gives a normal form for covariants modulo the ideal J. We call it the PQ-Theorem.

8.1 PQ-Theorem. *Every covariant φ is equivalent modulo J to a linear combination of symbolic expressions of the form PQ where Q is a product of symbolic expressions of degree ≤ 2 and P has the form*

$$P = (ab)^\mu(bc)^\nu(cd)^{\mu_1}(de)^{\nu_1}\cdots a_x^{p_a-\mu}\, b_x^{p_b-\mu-\nu}\, c_x^{p_c-\nu-\mu_1}\cdots$$

where

$$\begin{array}{ll} \mu < \frac{5}{8}N & 0 < \nu \leq \frac{1}{2}\mu \\ \mu_1 < \mu - \nu & 0 < \nu_1 \leq \frac{1}{2}\mu_1 \\ \vdots & \vdots \\ \mu_i < \mu_{i-1} - \nu_{i-1} & 0 < \nu_i \leq \frac{1}{2}\mu_i \\ \vdots & \vdots \end{array}$$

8.2 Remark. It follows from the definition of the ideal J that the factors of Q of degree 2 have order $> N_1$ and $\leq 2N-2$. Moreover, for $N \leq 3$ the factors P can be omitted: Every covariant modulo J is a linear combination of products of symbolic expressions of degree ≤ 2.

Proof of the PQ-Theorem. (0) The theorem is clear for $N=1$. It follows from the First Fundamental Theorem 3.1 that the covariants of $\mathcal{A}/J$ are generated by the covariants in degree 1, hence are linear combinations of symbolic expressions of type Q. Thus we can use induction and assume that $N \geq 2$.

(1) Let A be a symbolic expression of degree d and category λ. The claim is obvious for symbolic expressions of degree ≤ 2 and also for those of large category. In fact, $\operatorname{cat} A \geq \frac{5}{8}N$ implies that $A \in I \subset J$ (see §6 Example 1). Thus we can assume that the claim holds for all symbolic expressions of degree $< d$ and for those of degree d and of category $> \lambda$ (and that $\lambda < \frac{5}{8}N$).

(2) Let $(ab)^\lambda$ be a 2-factor of A with maximal exponent $\lambda = \operatorname{cat} A$. Then A is a ρ-contraction of $K_{ab}^\lambda = (ab)^\lambda a_x^{p_a-\lambda} b_x^{p_b-\lambda}$ with a symbolic expression B of degree $d-2$. By induction, we have $B = \sum q_i P_i Q_i$ where P_i and Q_i satisfy the conditions of the theorem. If $\rho = 0$ then $A = K_{ab}^\lambda B = \sum q_i P_i (K_{ab}^\lambda Q_i)$ and the claim follows. So we can assume that the theorem holds for any ρ'-contractions of K_{ab}^λ with any B' of degree $< d$ where $\rho' < \rho$.

(3) As in the proofs of Proposition 6.8 in §6 and Propositions 7.1 and 7.2 in §7 it follows, using Lemma A and C of §5 and induction, that the following statements are equivalent:

(i) *The theorem holds for one ρ-contraction of K_{ab}^λ with B.*
(ii) *The theorem holds for all ρ-contractions of K_{ab}^λ with B.*
(iii) *The theorem holds for the transvection $[K_{ab}^\lambda, B]_\rho$.*

In fact, (ii) implies (i) and, by Lemma 5.3.A, also (iii). Conversely, if T is an arbitrary ρ-contraction of K_{ab}^λ with B then, by Lemma 5.5.C,

$$T - [K_{ab}^\lambda, B]_\rho = \sum p_i [K_{ab}^{\lambda_i}, B_i]_{\rho_i}$$

where $\lambda_i \geq \lambda$ and $\rho_i < \rho$. Thus, either $\lambda_i > \lambda$ and then $[K_{ab}^{\lambda_i}, B_i]_{\rho_i}$ is a sum of symbolic expression of category $> \lambda$, or $\lambda_i = \lambda$ and then $[K_{ab}^{\lambda_i}, B_i]_{\rho_i}$ is a sum of lower contractions of K_{ab}^λ with B_i. In both cases, induction applies and so the right hand side of the equation above satisfies the conditions of the theorem. This shows the equivalence of (iii) and (i). Since T was arbitrary, we also get that (iii) implies (ii).

It follows now that it is suffices to prove the claim for one ρ-contraction of K_{ab}^{λ} with a product PQ where P and Q satisfy the conditions of the theorem.

(4) Write $P = (cd)^{\mu_1}(de)^{\nu_1}\cdots$ where $\mu_1 \le \frac{5}{8}N$, $\nu_1 \le \frac{1}{2}\mu_1, \dots$. (In case $P = 1$ we have $Q = (cd)^{\mu_1}\cdots$ where $\mu_1 \le \frac{5}{8}N$ and the same arguments can be used.) Moreover, we can assume that $p_b \ge p_a$. We now choose the ρ-contraction A of K_{ab}^{λ} with PQ in such a way that the exponent ν of the bc-factor becomes maximal. This means that $\nu = \min(p_b - \lambda, p_c - \mu_1, \rho)$.

(4a) If $\nu = p_c - \mu_1$ then we get $\nu + \mu_1 = p_c > \frac{3}{4}N > \frac{5}{8}N > \lambda$. This inequality shows that for $m :=$ the sum of the exponents of the $abcd$-term of A we get

$$m \ge \lambda + \nu + \mu_1 > 2\lambda.$$

By Proposition 7.2 this shows that A is equivalent modulo J to a linear combination of symbolic expressions of category $> \lambda$, and we are done.

(4b) If $\nu = p_b - \lambda$ then we look at the abc-term and find

$$\lambda < \frac{3}{4}N < p_a + p_b - 2\lambda \le 2(p_b - \lambda) = 2\nu,$$

since otherwise A belongs to J. Hence, $3\lambda < 2(\lambda + \nu) \le 2n$ where $n :=$ the sum of the exponents of the abc-term of A. It follows from Proposition 6.8 that A is equivalent modulo J to a linear combination of symbolic expressions of category $> \lambda$, and we are again done.

(4c) Finally, if $\nu = \rho$ then $A = (ab)^{\lambda}(bc)^{\nu}(cd)^{\mu_1}(de)^{\nu_1}\cdots$. If $\lambda < 2\nu$ or $\lambda < \nu + \mu_1$ it follows again from Proposition 6.8 or from Proposition 7.2 that A is equivalent modulo J to a linear combination of symbolic expressions of category $> \lambda$.

This completes the proof of the PQ-Theorem. □

In the proof of the RST-Theorem 2.6 we will need an estimate for the degree and the order of the covariants of type P.

8.3 Corollary. *For the covariants of type P we have*

$$\deg P \le \frac{5}{4}N \quad \textit{and} \quad \operatorname{ord} P + \deg P \le \frac{55}{64}N^2.$$

Proof. Consider a covariant of type P

$$P = (ab)^{\mu}(bc)^{\nu}(cd)^{\mu_1}(de)^{\nu_1}\cdots a_x^{p_a-\mu}\, b_x^{p_b-\mu-\nu}\, c_x^{p_c-\nu-\mu_1}\cdots$$

where

$$\begin{array}{ll} \mu < \frac{5}{8}N & 0 < \nu \le \frac{1}{2}\mu \\ \mu_1 < \mu - \nu & 0 < \nu_1 \le \frac{1}{2}\mu_1 \\ \vdots & \vdots \\ \mu_i < \mu_{i-1} - \nu_{i-1} & 0 < \nu_i \le \frac{1}{2}\mu_i \\ \vdots & \vdots \end{array}$$

Such a covariant P only exists for $N \ge 3$, and we can even assume $N > 3$, because for $N = 3$ we have $P = (ab)$ and the claim is obvious. Since the sequence $\mu_0 := \mu, \mu_1, \mu_2, \dots, \mu_k$ is decreasing, the number of symbols in P (which equals $\deg P$) is $\le 2\mu < \frac{5}{4}N$. In order to estimate the order of P we can assume that $\mu_k = 1$ (otherwise we add $\nu_k = 1$ and $\mu_{k+1} = 1$ which increases the order and the degree). Moreover, we can assume that all $\nu_i = 1$ and all $p_j = N$. Then we find

$$\operatorname{ord} P = (2k+2)N - 2(\sum_{i=0}^{k} \mu_i + k).$$

For fixed degree $\deg P = 2k + 2$ the maximal order is obtained by putting $\mu_k = 1, \mu_{k-1} = 2, \mu_{k-2} = 3, \dots, \mu_1 = k, \mu_0 = k + 1$. This gives

$$\operatorname{ord} P = (2k+2)N - 2(\sum_{j=1}^{k+1} j + k) = (2k+2)N - (k^2 + 5k + 2).$$

Denote this polynomial by $f(k)$. The maximal value of $f(k)$ is obtained for $k = \frac{2N-5}{2}$. Since $\mu = k + 1 = \frac{2N-3}{2} > \frac{5}{8}N$ we finally get

$$\operatorname{ord} P + \deg P \le f(\frac{5}{8}N - 1) + \frac{5}{4}N = \frac{55}{64}N^2 - \frac{5}{8}N + 2 < \frac{55}{64}N^2.$$

□

§9. Proof of the RST-Theorem

Let us shortly recall the RST-Theorem from §2.

9.1 RST-Theorem. *Every covariant in the algebra $\mathcal{B}^U$ is a linear combination of products RST where the factors R, S and T have the following form:*

- *R is a covariant of order $< 2N^2$ whose sum of order and degree is $< 9N^2$;*
- *S is a product of factors of degree 1 (and order $\le N$) or of degree 2 and order at least 2 and at most $2N - 2$;*
- *T is a product of invariants of degree $\le 7N - 5$.*

The algebra B is defined to be

$$\mathcal{B} = A_0/J_0 \otimes_K A_1/J_1 \otimes \ldots \otimes_K A_\tau = (\mathcal{O}(W)/J) \otimes \mathcal{B}_1$$

where $\mathcal{B}_1$ is defined as $\mathcal{B}$ starting with $W_1 := (J/\mathfrak{m}J)^*$ instead of W. The result will be obtained from this decomposition by induction using the description of the covariants of $\mathcal{O}(W)/J$ given in the PQ-Theorem 8.1.

Proof. We prove the theorem by induction on N. The case $N = 1$ is clear by the First Fundamental Theorem 3.1: In this case we have $\tau = 0$ and $\mathcal{B} = \mathcal{A}$, and the generating covariants are either of degree 1, hence of type S, or invariants of degree 2, hence of type T.
For a given $N \geq 2$ we use induction on the degree of the covariant, the degree 1 case again being clear since every covariant of degree 1 is of type S.

By induction on N we can assume that each covariant in B_1 can be written as a linear combination of products $R_1S_1T_1$ where

(1) R_1 is a covariant of order $< 2N_1^2$, whose sum of order and degree is $< 9N_1^2$;
(2) S_1 is a product of factors of degree 1 (and order $\leq N_1$ and ≥ 1) and of degree 2 and order at least 2 and at most $2N_1 - 2$;
(3) T_1 is a product of invariants of degree $\leq 7N_1 - 5$.

By the PQ-Theorem 8.1 we can assume that every covariant in A/J can be written as a linear combination of products PQ where

(1) Q is a product of covariants of degree 1 or 2;
(2) P is a covariant whose symbolic expression is of the type

$$P = (ab)^\mu (bc)^\nu (cd)^{\mu_1} (de)^{\nu_1} \cdots a_x^{p_a-\mu}\, b_x^{p_b-\mu-\nu}\, c_x^{p_c-\nu-\mu_1} \cdots$$

where

$$\begin{array}{ll} \mu < \frac{5}{8}N & 0 < \nu \leq \frac{1}{2}\mu \\ \mu_1 < \mu - \nu & 0 < \nu_1 \leq \frac{1}{2}\mu_1 \\ \vdots & \vdots \\ \mu_i < \mu_{i-1} - \nu_{i-1} & 0 < \nu_i \leq \frac{1}{2}\mu_i \\ \vdots & \vdots \end{array}$$

Recall that every factor of Q of degree 2 has order $> N_1$ and $\leq 2N - 2$ (Remark 8.2). Moreover, by Corollary 1 of §8, we have for $o := \operatorname{ord} P$ and $d := \deg P$

$$d \leq \frac{5}{4}N \quad \text{and} \quad o + d \leq \frac{55}{64}N^2.$$

Every covariant in $\mathcal{B}^U$ is a linear combination of the transvections

$$[PQ, R_1S_1T_1]_\lambda$$

(see Proposition 5.7). We have to show that every such transvection is a linear combination of products RST satisfying the conditions of the theorem. We can assume that $T_1 = 1$. Indeed, T_1 is an invariant and so our transvection is a product $T_1 \cdot [PQ, R_1S_1]_\lambda$. By induction, $[PQ, R_1S_1]_\lambda$ is a linear combination of products RST and we can include T_1 as an additional factor in T.

Now we proceed by induction on λ. If $\lambda = 0$ we are dealing with the product $PQ \cdot R_1S_1 = (PR_1)(QS_1)$. It is clear that QS_1 is a product of type S. We will show that PR_1 is a covariant of type R. Denoting by d_1, o_1 the degree and the order of R_1 we get

$$\operatorname{ord} PR_1 = o + o_1 < \frac{55}{64}N^2 + 2N_1^2 < 2N^2,$$
$$\operatorname{ord} PR_1 + \deg PR_1 = (o + d) + (o_1 + d_1) < \frac{55}{64}N^2 + 9N_1^2 < 9N^2.$$

Thus we can assume that the theorem is true for all covariants of smaller degree than δ and for all λ'-transvections of degree δ with $\lambda' < \lambda$. By Lemma 5.5.C it is enough to prove the theorem for one λ-contraction U of PQ with R_1S_1 which can be chosen arbitrarily.

We will denote the degrees and orders of the covariants involved as indicated in the following table:

covariant	order	degree
P	o	d
Q	ω	δ
R_1	o_1	d_1
S_1	ω_1	δ_1

From the considerations above we can deduce that these degrees and orders satisfy the following inequalities:

$$d < \frac{5}{4}N \qquad o < \frac{55}{64}N^2 \qquad o + d < \frac{55}{64}N^2$$
$$o_1 < 2N_1^2 < \frac{9}{8}N^2 \qquad o_1 + d_1 < 9N_1^2 < \frac{81}{16}N^2$$

We also have $\omega \geq \delta$ and $\omega_1 \geq \delta_1$. The first inequality follows because each factor of Q of degree 2 has order $> N_1 \geq 1$, and the second is clear by assumption:

Finally, we denote o' and d' order and degree of the λ-contraction U of PQ with R_1S_1 . Obviously, we have

$$o' = o + \omega + o_1 + \omega_1 - 2\lambda$$
$$o' + d' = o + \omega + o_1 + \omega_1 - 2\lambda + d + \delta + d_1 + \delta_1$$

We subdivide our consideration into four cases. The first three are rather special, the last one is the main case.

I: Assume $Q = S_1 = 1$. Then U is of type R. In fact, we have

$$o' = o + o_1 - 2\lambda < \frac{55}{64}N^2 + \frac{9}{8}N^2 < 2N^2$$
$$o' + d' = o + d + o_1 + d_1 - 2\lambda < \frac{55}{64}N^2 + \frac{81}{16}N^2 < 9N^2$$

II: Assume $S_1 = 1$ and $Q \neq 1$. Let L be a factor of Q of order $\omega_L \leq 2N - 2$, $Q = LQ'$. If $\lambda \leq o + \omega - \omega_L$ then we can choose the contraction U without using the symbols of L, i.e. $U = LU'$ where U' is a λ-contraction of PQ' with R_1. By induction, U' can be written as a linear combination of products RST and we can add L as an additional factor to S.

So we can therefore assume that $\lambda > o + \omega - \omega_L$. Then we claim that U is of type R. In fact,

$$\begin{aligned} o' &< o + \omega + o_1 - 2(o + \omega - \omega_L) = o_1 + \omega_L - (\omega - \omega_L) - o \\ &< \frac{9}{8}N^2 + (2N - 2) < 2N^2, \end{aligned}$$

$$\begin{aligned} o' + d' &< o + \omega + o_1 - 2(o + \omega - \omega_L) + d + \delta + d_1 \\ &< (o_1 + d_1) + d + 2\omega_L - o - (\omega - \delta) \\ &< \frac{81}{16}N^2 + \frac{5}{4}N + 2(2N - 2) < 9N^2. \end{aligned}$$

(Here we used $\omega \geq \delta$.)

III: Assume $S_1 \neq 1$ and $Q = 1$. Let M be a factor of S_1 of degree 1 or 2 and of order $\omega_M \leq \max(N_1, 2N_1 - 2) < 2N_1$, $S_1 = MS_1'$. If $\lambda \leq o_1 + \omega_1 - \omega_M$ we can choose the contraction U without using the symbols of M, i.e. $U = MU'$ where U' is a contraction of P with R_1S_1. Then we finish the proof as in Case II.

Thus we can assume that $\lambda > o_1 + \omega_1 - \omega_M$. Now we show that U is of the type R:

$$\begin{aligned} o' &< o + o_1 + \omega_1 - 2(o_1 + \omega_1 - \omega_M) \\ &< o + \omega_M - (\omega_1 - \omega_M) < \frac{55}{64}N^2 + \omega_M < \frac{55}{64}N^2 + 2N_1 < 2N^2, \end{aligned}$$

$$\begin{aligned} o' + d' &< o + \omega_1 + o_1 - 2(o_1 + \omega_1 - \omega_M) + d + \delta_1 + d_1 \\ &< (o + d) + d_1 + 2\omega_M - (o_1 - \delta_1) \\ &< \frac{55}{64}N^2 + \frac{81}{16}N^2 + 2\omega_M < 9N^2. \end{aligned}$$

IV: It remains the case where $S_1, Q \neq 1$. Let us write $Q = LL_1 \dots L_m$ and $S_1 = MM_1 \dots M_n$ as products of factors of degree 1 and 2. We may assume that L and M have maximal order among all factors of Q (resp. S_1). Let $\rho := \operatorname{ord}(L)$, $\rho_1 := \operatorname{ord}(M)$. If $\lambda \leq \max(o + \omega - \rho, o_1 + \omega_1 - \rho_1)$ then, as in case II and III, we choose the contraction U in such way that it has the factor M or L and finish the argument by induction. Thus we can assume that $\lambda > \max(o + \omega - \rho, o_1 + \omega_1 - \rho_1) \geq \max(\omega - \rho, \omega_1 - \rho_1)$.

IVa: Assume that $m \geq \rho_1$ and $n \geq \rho$. We want to show that the contraction U can be chosen in such a way that it contains an invariant of type T. For this consider the products

$$LL_1L_2 \cdots L_{\rho_1}, \quad L_1L_2 \cdots L_{\rho_1}, \quad L_2 \cdots L_{\rho_1}, \quad \dots, \quad L_{\rho_1}.$$

Their respective orders $\mu_0, \dots, \mu_{\rho_1}$ form a decreasing sequence where the difference of consecutive terms is $\leq \rho$. Similarly, consider the products

$$MM_1M_2 \cdots M_\rho, \quad M_1M_2 \cdots M_\rho, \quad M_2 \cdots M_\rho, \quad \dots, \quad M_\rho.$$

Their respective orders $\nu_0, \dots, \nu_\rho$ also form a decreasing sequence where the difference of consecutive terms is $\leq \rho_1$ and $\nu_0 \geq \rho + 1 =$ the number of factors.

Assume $\nu_0 \leq \mu_0$. (The case $\mu_0 \leq \nu_0$ can be handled similarly.) Then all numbers ν_j, $j = 1, \dots, \rho$, are smaller than μ_0. Let μ_{ν_j} be the last member of the sequence of the μ_i's which is $\geq \nu_j$ and consider the ρ differences $\mu_{\nu_j} - \nu_j$, $j = 1, 2, \dots, \rho$. They form ρ non-negative numbers, all of which are $< \rho$. Indeed, for each j we have

$$\mu_{\nu_j} - \nu_j < \mu_{\nu_j} - \mu_{\nu_j+1} \leq \rho.$$

Therefore, either one of the differences is 0 or two of them are equal. If $\mu_{\nu_j} - \nu_j = 0$ for some $j > 0$, then the products $L_{\nu_j}L_{\nu_j+1} \cdots L_{\rho_1}$ and $M_jM_{j+1} \cdots M_\rho$ have the same order. If $\mu_{\nu_j} - \nu_j = \mu_{\nu_k} - \nu_k$ for some $j > k > 0$, then the products $L_{\nu_k} \dots L_{\nu_j}$ and $M_k \dots M_j$ have the same order. Call this order π. We have $\pi \leq \omega_1 - \rho_1$ because in both products of the M_i's the factor M is missing. Hence $\pi < \lambda$.

This implies that we can choose the λ-contraction U of PQ with R_1S_1 in such way that the two subproducts of the same order π are completely contracted to an invariant factor I of U. It's degree is the sum of degrees of the two subproducts. Since the first subproduct contains at most $\rho_1 + 1$ factors, the second at most ρ, and every factor has degree 1 or 2 we obtain $\deg I \leq 2(\rho + \rho_1 + 1)$. Now $\rho \leq 2N - 2$ and $\rho_1 \leq \max(N_1, 2N_1 - 2)$ by definition. If $N > 2$ and so $N_1 > 1$ this gives

$$\deg I \leq 2(2N - 2 + 2N_1 - 2 + 1) \leq 7N - 6.$$

If $N = 2$ and so $N_1 = 1$ and $\rho_1 = 1$ we find

$$\deg I \leq 2(2 + 1 + 1) = 8 = 7N - 6.$$

Therefore, I is a factor of type T. Induction on degree concludes the proof.

IVb: Assume that $m \leq \rho_1 - 1$. We want to show that the contraction U is of type R, i.e., that $o' < 2N^2$ and that $o' + d' < 9N^2$.

Since $Q = LL_1 \ldots L_m$, we have $\omega \leq \rho\rho_1$ and $\delta \leq 2\rho_1$. Moreover, we have seen above that $\lambda > \max(o + \omega - \rho, o_1 + \omega_1 - \rho_1)$. Hence

$$o' = o + \omega + o_1 + \omega_1 - 2\lambda < \rho + \rho_1.$$

For $N = 2$ we have $\rho \leq 2$, $\rho_1 \leq 1$ and so $o' < 3 < 2N^2$. For $N > 2$ we have $\rho \leq 2N - 2$, $\rho_1 \leq 2N_1 - 2 \leq \frac{3}{2}N - 2$. This gives $o' \leq \frac{7}{2}N - 4 < 2N^2$, hence

$$o' < \rho + \rho_1 < 2N^2$$

On the other hand we find

$$\begin{aligned} o' + d' &= o + \omega + o_1 + \omega_1 - 2\lambda + d + \delta + d_1 + \delta_1 \\ &< o + \omega + o_1 + \omega_1 - 2(o_1 + \omega_1 - \rho_1) + d + \delta + d_1 + \delta_1 \\ &< (o + d) + (\omega + 2\rho_1 + \delta) + d_1 - (\omega_1 - \delta_1) \\ &< \frac{55}{64}N^2 + (\rho + 4)\rho_1 + \frac{81}{16}N^2 \end{aligned}$$

For $N > 2$, we have $\rho \leq 2N - 2$ and $\rho_1 \leq 2N_1 - 2$. Therefore

$$(\rho + 4)\rho_1 \leq (2N + 2)(2N_1 - 2) < 4NN_1 \leq 3N^2$$

and this also holds for $N = 2$. Therefore

$$o' + d' < (\frac{55}{64} + 3 + \frac{81}{16})N^2 < 9N^2.$$

This shows that U is of type R.

IVc: Assume finally that $n \leq \rho - 1$. Again this will imply that U is of type R.

Since $S_1 = MM_1 \cdots M_n$ we have $\omega_1 \leq \rho\rho_1$ and $\delta_1 \leq 2\rho_1$. As in the previous case we find

$$o' < \rho + \rho_1 \leq 2N^2$$

and

$$\begin{aligned} o' + d' &< (o_1 + d_1) + (\omega_1 + 2\rho + \delta_1) + d - (\omega - \delta) \\ &< \frac{81}{16}N^2 + \rho(\rho_1 + 4) + \frac{5}{4}N < 9N^2. \end{aligned}$$

The finishes the proof of the RST-Theorem. □

§10. The action of GL_m and the type of a covariant

In the following section we will use Jordan's method to determine a minimal generating system for the covariants of (several copies of) binary quadratics and binary cubics. For that purpose let us first make some general remarks.

10.1 The action of GL_m. Let $W := M \otimes V_N$ where M is an arbitrary vector space of dimension m. Choosing a basis $\{u_1, u_2, \dots , u_m\}$ we can identify W with V_N^m. Clearly, the group $\mathrm{GL}(M) \times \mathrm{SL}_2$ acts on W and therefore also on the coordinate ring $\mathcal{O}(W)$. From Cauchy's formula (see [GW98, Cor. 4.5.19]) we get the decomposition

$$\mathcal{O}(W) = \bigoplus_{\mathrm{ht}\,\mu \le N+1} S_\mu(M^*) \otimes S_\mu(V_N^*) \tag{1}$$

where μ runs through the partitions of height $\le \dim V_N = N + 1$ and S_μ denotes the corresponding Schur functor (cf. [ABW82]; we assume that $\dim M$ is large, i.e. $\ge N + 1$.) Moreover, we have $S_\mu(M^*) \otimes S_\mu(V_N^*) \subset \mathcal{O}(W)_{|\mu|}$ where $|\mu| := \mu_1 + \mu_2 + \mu_3 + \cdots$ is the length of the partition. We can further decompose the SL_2 representation $S_\mu(V_N^*)$ using the formulas of Jacobi-Trudi and Sylvester (see [Sch68, §7 Satz 2.22]) or the computer program LiE [Lie92].

It follows that a minimal system of generators for the covariants of W can be chosen to be a direct sum of subspaces of the form $S_\mu(M^*) \otimes V_e \subset \mathcal{O}(W)_d$. Thus, a generating covariant for W of order e and degree d always comes coupled with an irreducible representation S_μ of $\mathrm{GL}(M)$ where $|\mu| = d$.

For our purpose, it is more convenient to pass to the dual partition μ' of μ and to use the notation $S^{\mu'}$ instead of S_μ. Thus S^k is the k-th symmetric power and $S^{1,1,\dots,1} = \bigwedge^k$ is the k-th exterior power.

10.2 The type of a covariant. There is another important point here, relating the considerations above to the symbolic method and to what is classically called the *type* of a covariant. Consider a symbolic expression

$$P = P_{\lambda,\sigma} = \prod_{a \ne b} (ab)^{\lambda_{a,b}} \prod_c c_x^{\sigma_c}$$

in the symbols $a, b, c, \dots$ from the set $\mathcal{J}$ where every symbol has weight N, i.e., it appears N times in P. In order to define a covariant φ_P corresponding to P (shortly, *a covariant of type* P) we have to associate to every symbol a occurring in P a copy of V_N in $W = V_N^m$. (Then the sum e of the exponents σ_c in P is equal to the order of the covariant and the number d of different symbols in P is equal to the degree of the covariant, see §4).

More generally, we can choose for every symbol a an (non-zero) element $u_a \in M$ to obtain a covariant of $W = M \otimes V_N$ of degree d and order e in the obvious way. Doing this in all possible ways the corresponding covariants linearly span a subspace C_P of $\mathcal{O}(M \otimes V_N)_d$ of the form $C_P = T_P \otimes V_e$ where T_P is a (not necessarily irreducible) representation of $\mathrm{GL}(M)$. We will express this by saying that $T_P \otimes V_e$ *are the covariants of type* P or that *the covariants of type* P *form the subspace* $T_P \otimes V_e$.

The classical results describe a "minimal" generating set for the covariants in terms of these types, i.e. by giving the symbolic expressions. We will complete this by describing the *irreducible types*, i.e., the spaces $T' \otimes V_e$ where $T' \subset T_P$ are the irreducible subspaces which are needed to form a minimal generating system.

There is a nice method due to BURCKHARDT [Bu98] which describes the subspace T_P associated to a symbolic expression P. For this we consider the obvious linear action of the summetric group S_n ($n = |\mathcal{J}|$) on the symbolic algebra $\mathrm{Symb}_{\mathcal{J}}$ (3.4). In the following statement we use the SCHUR-duality between irreducible representations of the symmetric group and the general linear group: The irreducible representations L^μ of S_n are parametrized by partitions μ of length $|\mu| = n$ and thus give rise to irreducible representations $S^\mu(M)$ (and $S^\mu(M^*)$) of $\mathrm{GL}(M)$.

10.3 Proposition. *Let* P *be a symbolic expression in the letters* $a \in \mathcal{J}$ *and denote by* $S_n P \subset \mathrm{Symb}_{\mathcal{J}}$ *the* S_n*-stable subspace spanned by* P. *If* $S_n P = \bigoplus_i L^{\mu_i}$ *is the decomposition into irreducible components then* $T_P = \bigoplus_i S^{\mu_i}(M^*)$.

§11. The Covariants of binary quadratics

In this section and the following we use JORDAN's method to determine a minimal generating system for the covariants of (several copies of) binary quadratics and binary cubics. These results are classical and can be found in the book [GY03] of GRACE and YOUNG. However, in the case of cubics we will give a more precise statement using representation theory of the general linear group.

11.1 Binary quadratics ($N = 2$). Take $W := M \otimes V_2$ where M is a vector space of dimension $m \geq 3$. According to JORDAN's method we have to look at the ideal $J \subset \mathcal{O}(W)$ generated by all covariants of degree ≤ 3 and order $\leq N_1 = \lfloor \frac{3}{4} N \rfloor = 1$ and describe the covariants of $\mathcal{O}(W)/J$. Since all covariants of W are of even order (the center $\pm E$ acts trivially!) the ideal J is generated by invariants.

Now we use the following decompositions of the coordinate ring $\mathcal{O}(W)$ in low degrees. They follow from CAUCHY's formula and the decompositions of the SCHUR functors $S^\lambda V_2$. (We have been using the program LiE [Lie92] to obtain these decompositions.)

$$\begin{aligned}
\mathcal{O}(W)_1 &= M^* \otimes V_2, \\
\mathcal{O}(W)_2 &= S^2 M^* \otimes (V_4 \oplus V_0) \bigoplus {\textstyle\bigwedge}^2 M^* \otimes V_2, \\
\mathcal{O}(W)_3 &= S^3 M^* \otimes (V_6 \oplus V_2) \bigoplus S^{2,1} M^* \otimes (V_4 \oplus V_2) \bigoplus {\textstyle\bigwedge}^3 M^* \otimes V_0, \qquad (1) \\
\mathcal{O}(W)_4 &= S^4 M^* \otimes (V_8 \oplus V_4 \oplus V_0) \bigoplus S^{3,1} M^* \otimes (V_6 \oplus V_4 \oplus V_2) \bigoplus \\
&\qquad S^{2,2} M^* \otimes (V_4 \oplus V_0) \bigoplus S^{2,1,1} M^* \otimes V_2.
\end{aligned}$$

It follows that the ideal J is generated by the invariants of type $(ab)^2$ in degree 2 and $(ab)(bc)(ca)$ in degree 3, corresponding to the subspaces $S^2M^* \otimes V_0$ and $\bigwedge^3 M^* \otimes V_0$ which form the isotypic components of V_0 in degree 2 and 3.

11.2 Lemma.

(a) *The ideal J contains all invariants. It is generated by the invariants of type $(ab)^2$ and $(ab)(bc)(ca)$ corresponding to the subspaces $S^2M^* \otimes V_0$ and $\bigwedge^3 M^* \otimes V_0$ in degree 2 and 3.*

(b) *The covariants of $\mathcal{O}(M \otimes V_2)/J$ are generated by the covariants of type a_x^2 and $(ab)a_x b_x$ corresponding to the subspaces $M^* \otimes V_2$ and $\bigwedge^2 M^* \otimes V_2$ in degree 1 and 2. Moreover, $(\bigwedge^2 M^* \otimes V_2)^2 = (0)$ in $\mathcal{O}(M \otimes V_2)/J$.*

(c) *All covariants of degree 3 and order ≤ 2 and of degree 4 and order ≤ 4 belong to J, and we have*

$$\begin{aligned}(\mathcal{O}(M \otimes V_2)/J)_3 &= S^3M^* \otimes V_6 \bigoplus S^{2,1}M^* \otimes V_4,\\ (\mathcal{O}(M \otimes V_2)/J)_4 &= S^4M^* \otimes V_8 \bigoplus S^{3,1}M^* \otimes V_6.\end{aligned} \tag{2}$$

In Proposition 11.5 we will complete the picture by giving a description of the algebra $\mathcal{O}(M \otimes V_2)/J$ as a $\mathrm{GL}(M) \times \mathrm{SL}_2$-module. In order to determine a minimal system of generators in the following proposition we will only need item (a) and the first part of (b).

Proof. We have already shown the second part of (a). Looking at the PQ-Theorem 8.1 we see that the covariants of $\mathcal{O}(W)/J$ are generated by covariants of degree ≤ 2, because the only covariants of the form P are of type $(ab)a_x b_x$ (see Remark 8.2). Thus we get the first part of assertion (b). In particular, $\mathcal{O}(W)/J$ doesn't contain any invariants and so the first part of (a) follows, too.

For the second part of (b) we remark that $(\bigwedge^2 M^* \otimes V_2)^2$ consists of covariants of degree 4 and order ≤ 4. We will see below that these covariants all belong to J (see the second formula in item (c)).

For (c) we look again at the decompositions (1). From the abc-Theorem 6.7 we see that every covariant of degree 3 and order ≤ 2 belongs to J. In degree 4, Proposition 7.1 implies that modulo J every covariant is a linear combination of covariants of types $a_x^2 b_x^2 c_x^2 d_x^2$, $(ab)a_x b_x c_x^2 d_x^2$ and $(ab)^2 d_x^2 e_x^2$. The last one belongs to J and the second corresponds to the subspace $S^{3,1}M^* \otimes V_6$, because this is the isotypic component of V_6 in degree 4. This gives the first part of (c), and the second follows, because J, being generated by invariants of degree ≥ 2, can only contain covariants of order $\leq 2d$ in degree $d+2$. □

11.3 Proposition. *A minimal generating system for the covariants of binary quadratics $V_2^m (= M \otimes V_2)$ is given by the covariants of type a_x^2 and $(ab)a_x b_x$, and the* *invariants of type* $(ab)^2$ *and* $(ab)(bc)(ca)$*. They correspond to the following subspaces in* $\mathcal{O}(M \otimes V_2)$*:*

$$\begin{aligned} a_x^2 : \quad & M^* \otimes V_2 \subset \mathcal{O}(M \otimes V_2)_1 \\ (ab)a_x b_x : \quad & \textstyle\bigwedge^2 M^* \otimes V_2 \subset \mathcal{O}(M \otimes V_2)_2 \\ (ab)^2 : \quad & S^2 M^* \otimes V_0 \subset \mathcal{O}(M \otimes V_2)_2 \\ (ab)(bc)(ca) : \quad & \textstyle\bigwedge^3 M^* \otimes V_0 \subset \mathcal{O}(M \otimes V_2)_3 \end{aligned}$$

Proof. The following general fact is well-known and easy to prove: If $I \subset A$ is an ideal generated by invariants, then we obtain a generating set for the covariants of A by taking a generating set of the ideal I consisting of invariants together with the liftings of a generating set of the covariants of A/I to A.

Now it follows from Lemma 11.2 (a) and (b) that the given set generates the covariants and that the corresponding subspaces have the given form. It is obvious that the set is minimal. □

11.4 Remarks. (a) If we denote by F a covariant of type a_x^2, then the other generators listed in the proposition are given by the following transvections:

$$F, \qquad [F,F]_1, \qquad [F,F]_2, \qquad [[F,F]_1,F]_2.$$

This is clear because the corresponding isotypic components in degree 1, 2 and 3 are irreducible (see the decompositions (1) in 11.1).

(b) The above results already show the strength of the general methods developed in §6–8, in particular of the PQ-Theorem, even in this "very small" case. Of course, there is also a direct argument for the first claim of the proposition, based on certain simple relations between symbolic expressions. In fact, the only indecomposable covariant of W of category 2 is $(ab)^2$. All others have category ≤ 1 and are therefore of the form $(ab)(bc)(cd)\cdots(fg)(ga)$ or $(ab)(bc)(cd)\cdots(fg)a_x g_x$. Now we have the following relations:

$$\begin{aligned} 2(ab)(bc)(cd)(da) &= (ab)^2(cd)^2 + (bc)^2(ad)^2 - (ac)^2(bd)^2, \\ 2(ab)(bc)(cd)(de) &= (bc)(cd)(db)(ae) - (cd)^2(ab)(be) - \\ &\qquad (bc)^2(ad)(de) + (bd)^2(ac)(ce). \end{aligned} \tag{3}$$

They show that all symbolic expressions are reducible if they contain at least 4 symbols, i.e., are of degree ≥ 4. Thus the only possible candidates for the generating system are those given in the proposition.

An interesting observation is the geometric interpretation of the ideal J. Recall that the *nullcone* $\mathcal{N}$ of $W = {V_2}^m$ is defined by the vanishing of all homogeneous invariants of positive degree. One knows that $\mathcal{N}$ is the set of all m-tuples of quadratics which are all squares of the same linear form, up to a scalar (see 4.4 Example (7)). This can also be seen by using the Hilbert Criterion (cf. [Kr85, III.2]).

11.5 Proposition. *The radical $\sqrt{J}$ is generated by all invariants of positive degree together with the covariants of type $(ab)a_x b_x$. The ideal $\sqrt{J}$ is prime and its zero set is the nullcone $\mathcal{N} \subset W$. Moreover, we have the following decompositions:*

$$\mathcal{O}(M \otimes V_2)/\sqrt{J} \simeq \bigoplus_{j \geq 0} S^j M^* \otimes V_{2j} \quad \textit{and}$$
$$\mathcal{O}(M \otimes V_2)/J \simeq \bigoplus_{j \geq 0} \left(S^j M^* \otimes V_{2j} \,\oplus\, S^{j-1,1} M^* \otimes V_{2j-2}\right).$$

Proof. Denote by I the ideal generated by J and $\bigwedge^2 M^* \otimes V_2$. It follows from Lemma 11.2 (b) that $I^2 \subset J$, because $(\bigwedge^2 M^* \otimes V_2)^2 \subset J$, and that the covariants of $\mathcal{O}(W)/I$ are generated by $M^* \otimes V_2$. Therefore, $\mathcal{O}(W)/I \simeq \bigoplus_{j\geq 0} S^j M^* \otimes V_{2j}$ and $(\mathcal{O}(W)/I)^U \simeq S(M^*)$. Moreover, both algebras are integral domains. This is obvious for the second. For the first we simply remark that every $\mathrm{GL}(M) \otimes \mathrm{SL}_2$-stable ideal $\neq (0)$ in $\mathcal{O}(W)/I$ is of finite codimension since it has to contain one of the irreducible subspaces $S^j M^* \otimes V_{2j}$. It follows that $I = \sqrt{J}$, that $\sqrt{J}$ is prime and that $\mathcal{O}(W)/\sqrt{J} \simeq \bigoplus_{j\geq 0} S^j M^* \otimes V_{2j}$. It remains to prove the second decomposition formula.

By Lemma 11.2 (b) the covariants of $\mathcal{O}(W)/J$ are generated by $M^* \otimes V_2$ and $\bigwedge^2 M^* \otimes V_2$, and $(\bigwedge^2 M^* \otimes V_2)^2 = (0)$ in $\mathcal{O}(W)/J$. Therefore, the only representations of SL_2 which can appear in degree j are V_{2j} and V_{2j-2}, and we have

$$(\mathcal{O}(M \otimes V_2)/J)_j = S^j M^* \otimes V_{2j} + S^{j-2} M^* \otimes V_{2j-4} \cdot \bigwedge^2 M^* \otimes V_2.$$

By Pieri's formula we have $S^{j-2}M^* \otimes \bigwedge^2 M^* \simeq S^{j-1,1}M^* \oplus S^{j-2,1,1}M^*$ (see [FH91, (6.9)] or [GW98, Cor. 9.2.4]). On the other hand, Cauchy's formula 10.1(1) gives

$$\mathcal{O}(M \otimes V_2)_j \simeq \bigoplus_{|\mu|=j} S^\mu M^* \otimes S^\mu V_2.$$

Since V_{2j-2} appears in $S^{j-1,1}V_2$, but not in $S^{j-2,1,1}V_2$ ($\subset S^{j-3}V_2$), we finally get that $S^{j-2}M^* \otimes V_{2j-4} \cdot \bigwedge^2 M^* \otimes V_2 = S^{j-1,1}M^* \otimes V_{2j-2}$, hence the claim. □

§12. The covariants of binary cubics

The next case we treat is the case of several cubics ($N = 3$). This can also be found in the book of Grace and Young ([GY03]). In [Sch87] Schwarz gave a rigorous proof of the description of the ring of invariants and their relations. Here we show that Jordan's approach is essentially the algorithm given by Grace and

YOUNG. Their proof will therefore be shown to be correct. As mentioned above, we are able to improve their result by using representation theory.

We take $W := M \otimes V_3$ where M is a vector space of dimension $m \geq 4$. For later references let us write down the decompositions of the $\mathcal{O}(W)$ in low degrees:

$$
\begin{aligned}
\mathcal{O}(W)_1 &= M^* \otimes V_3,\\
\mathcal{O}(W)_2 &= S^2 M^* \otimes (V_6 \oplus V_2) \bigoplus \bigwedge^2 M^* \otimes (V_4 \oplus V_0),\\
\mathcal{O}(W)_3 &= S^3 M^* \otimes (V_9 \oplus V_5 \oplus V_3) \bigoplus\\
&\quad S^{2,1} M^* \otimes (V_7 \oplus V_5 \oplus V_3 \oplus V_1) \bigoplus\\
&\quad \bigwedge^3 M^* \otimes V_3,\\
\mathcal{O}(W)_4 &= S^4 M^* \otimes (V_{12} \oplus V_8 \oplus V_6 \oplus V_4 \oplus V_0) \bigoplus\\
&\quad S^{3,1} M^* \otimes (V_{10} \oplus V_8 \oplus 2V_6 \oplus V_4 \oplus 2V_2) \bigoplus\\
&\quad S^{2,2} M^* \otimes (V_8 \oplus 2V_4 \oplus V_0) \bigoplus\\
&\quad S^{2,1,1} M^* \otimes (V_6 \oplus V_4 \oplus V_2) \bigoplus\\
&\quad \bigwedge^4 M^* \otimes V_0.
\end{aligned}
\tag{1}
$$

The formulae show that in degree 4 the representations $S^{3,1}M^* \otimes V_6$, $S^{3,1}M^* \otimes V_2$ and $S^{2,2}M^* \otimes V_4$ appear with multiplicity 2.

Following JORDAN's method we consider the ideal $J \subset \mathcal{O}(W)$ generated by all covariants of degree 2 and 3 and of order $\leq N_1 = \lfloor \frac{3}{4} N \rfloor = 2$. In degree 2, these are the covariants of type $(ab)^3$ and $(ab)^2 a_x b_x$ corresponding to the subspaces $\bigwedge^2 M^* \otimes V_0$ and $S^2 M^* \otimes V_2$. In degree 3, the only covariant of order ≤ 2 is of type $(ab)^2(ac)(bc)c_x$ corresponding to the subspaces $S^{2,1}M^* \otimes V_1$. However, by Example 6.6, every covariant of category ≥ 2 belongs to the ideal I generated by the covariants of type $(ab)^3$ and $(ab)^2 a_x b_x$, and so J is generated by these two types. This proves the first assertion of the following lemma.

12.1 Lemma.

(1) *The ideal J is generated by the covariants of type $H = (ab)^2 a_x b_x$ and the invariants of type $I = (ab)^3$ corresponding to the subspaces $S^2 M^* \otimes V_2$ and $\bigwedge^2 M^* \otimes V_0$ in degree 2. It contains all invariants.*

(2) *The covariants of $\mathcal{O}(W)/J$ are generated by those of type $F = a_x^3$ and $\Delta = (ab) a_x^2 b_x^2$ corresponding to the subspaces $M^* \otimes V_3$ and $\bigwedge^2 M^* \otimes V_4$ in degree 1 and 2.*

Proof. (1) We have already proved the first assertion. The second follows from (2) since $\mathcal{O}(W)/J$ doesn't contain any invariants.

(2) This is a consequence of the PQ-Theorem 8.1 which shows that every symbolic expression of type P has the form $P = (ab) a_x^2 b_x^2$ which is of degree 2 (see Remark 8.2). □

By Lemma 12.1(1) the subspace $E = S^2M^* \otimes V_2 \oplus \bigwedge^2 M^* \otimes V_0 \subset J$ is a minimal homogeneous and SL_2-stable subspace generating the ideal J. Put

$$W_1 := E^* = S^2M \otimes V_2 \,\oplus\, \bigwedge^2 M \otimes V_0.$$

We denote by H and I the linear covariant and invariant corresponding to the subspace $S^2M^* \otimes V_2 \subset \mathcal{O}(W_1)_1$ and $\bigwedge^2 M^* \otimes V_0 \subset \mathcal{O}(W_1)_1$. Then it follows from Proposition 11.3 and Remark 11.4(a) that the covariants of W_1 are generated by the invariants I, $I' := [H,H]_2$ and $I'' = [[H,H]_1, H]_2$, and the covariants H and $K := [H,H]_1$ corresponding to the following subspaces:

$$\begin{aligned}
I:&\quad \textstyle\bigwedge^2 M^* \otimes V_0 \subset \mathcal{O}(W_1)_1\\
I':&\quad S^2(S^2M^*) \otimes V_0 \subset \mathcal{O}(W_1)_2\\
I'':&\quad \textstyle\bigwedge^3 (S^2M^*) \otimes V_0 \subset \mathcal{O}(W_1)_3\\
H:&\quad S^2M^* \otimes V_2 \subset \mathcal{O}(W_1)_1\\
K:&\quad \textstyle\bigwedge^2 (S^2M^*) \otimes V_2 \subset \mathcal{O}(W_1)_2
\end{aligned}$$

Our next step is to determine the “irreducible” covariants of the tensor product $(\mathcal{O}(W)/J) \otimes \mathcal{O}(W_1)$. We know from our general approach in §2 that their images under the canonical homomorphism $\Phi\colon (\mathcal{O}(W)/J)\otimes\mathcal{O}(W_1) \to \bigoplus_i J^i/J^{i+1} = \operatorname{gr}_J \mathcal{O}(W)$ will generate the covariants of $\operatorname{gr}_J \mathcal{O}(W)$ and lift to a generating set for the covariants of $\mathcal{O}(W)$.

12.2 Lemma. *The covariants of $(\mathcal{O}(W)/J) \otimes \mathcal{O}(W_1)$ are generated by the covariants of $\mathcal{O}(W)/J$, the covariants of $\mathcal{O}(W_1)$ and the following transvections*

$$\begin{gathered}
[F,H]_1, \quad [F,H]_2, \quad [F,H_1H_2]_3, \quad [F_1F_2, H_1H_2H_3]_6,\\
[F,K]_1, \quad [F,K]_2, \quad [F,HK]_3, \quad [F_1F_2, H_1H_2K]_6,\\
[\Delta,H]_1, \quad [\Delta,H]_2, \quad [\Delta,H_1H_2]_3, \quad [\Delta,H_1H_2]_4,\\
[\Delta,K]_1, \quad [\Delta,K]_2, \quad [\Delta,HK]_3, \quad [\Delta,HK]_4,
\end{gathered} \tag{2}$$

where the symbols F and Δ denote the covariants of $\mathcal{O}(W)/J$ of type a_x^3 and $(ab)a_x^2b_x^2$, respectively, $W_1 = S^2M \otimes V_2 \oplus \bigwedge^2 M \otimes V_0$, and the symbols H denote the linear covariants of $\mathcal{O}(W_1)$ of order 2 (of type f_x^2) and $K = [H,H]_1$ the quadratic covariants of $\mathcal{O}(W_1)$ of order 2 (of type $(fg)f_xg_x$).

(We will see below that the covariants $[F,K]_1$, $[\Delta,H]_1$ and $[\Delta,K]_1$ are reducible and can therefore be deleted from the list above.)

Proof. We first recall that the covariants of the tensor product $\mathcal{O}(W)/J \otimes \mathcal{O}(W_1)$ are generated by the invariants of $\mathcal{O}(W)/J$, the invariants of $\mathcal{O}(W_1)$ and all transvections of the form

$$[F_1 \cdots F_\alpha \Delta_1 \cdots \Delta_\beta, H_1 \cdots H_\gamma]_\ell \quad \text{and} \quad [F_1 \cdots F_\alpha \Delta_1 \cdots \Delta_\beta, H_1 \cdots H_\gamma K]_\ell$$

where the symbols have the same meaning as in the Lemma. This follows from Proposition 5.7 and the fact that the covariants of $\mathcal{O}(W_1)$ are linearly spanned by the products $TH_1H_2 \cdots H_\gamma$ and $TH_1H_2 \cdots H_\gamma K$ where T is an invariant.

According to Proposition 5.7 we have to show that for all these transvections except those listed in the Lemma there is a decomposable ℓ-contraction of the two monomials $L := F_1F_2 \cdots F_\alpha \Delta_1 \Delta_2 \cdots \Delta_\beta$ and $R := H_1H_2 \cdots H_\gamma$ or $R := H_1H_2 \cdots H_\gamma K$.

If the left hand side L contains a factor Δ then $L = \Delta$. Otherwise, there is a decomposable contraction because all factors of the right hand side have order 2 (and Δ has order 4). In this case the right hand side R has at most two factors. This gives the last two lines of the list (1) in the Lemma.

Thus we can assume that $L = F_1F_2 \cdots F_\alpha$. Since H and K have order 2 it is clear that there is always a decomposabel contraction of L with $R := H_1H_2 \cdots H_\gamma$ and $R := H_1H_2 \cdots H_\gamma K$ in case L has three or more F-factors. Thus $L = F$ or $L = F_1F_2$, and it is easy to see that we are left with those given in the first two lines of the list (1) in the Lemma. □

12.3 Lifting covariants to $\mathcal{O}(W)$. We obtain a first set of generators for the covariants of the binary cubics $V_3^m = W$ by lifting the covariants listed in Lemma 12.2 to $\mathcal{O}(W)$. This lift is obvious for the generators F, Δ of $\mathcal{O}(W)/J$ and for the covariants H and I of $\mathcal{O}(W_1)$ since they are already given in symbolic form in Lemma 12.1 (we use the same symbol for the lift but with a tilde):

$$\tilde{F} = a_x^3, \quad \tilde{\Delta} = (ab)a_x^2b_x^2, \quad \tilde{H} = (ab)^2a_xb_x \quad \text{and} \quad \tilde{I} = (ab)^3.$$

They correspond to the subspaces $M^* \otimes V_3$, $\bigwedge^2 M^* \otimes V_4$, $S^2M^* \otimes V_2$ and $\bigwedge^3 M^* \otimes V_0$. The other generators are given as transvections. Therefore, we get a lift by forming the corresponding transvections in $\mathcal{O}(W)$, e.g.

$$[\tilde{H}, \tilde{H}]_1, \quad [\tilde{H}, \tilde{H}]_2, \quad [\tilde{F}, \tilde{H}]_1, \quad [\tilde{F}, \tilde{H}\tilde{K}]_3, \quad [\tilde{\Delta}, \tilde{H}_1\tilde{H}_2]_3, \quad \text{etc.}$$

On the other hand we know that we can always replace these transvections by arbitrary chosen contractions. In particular, we will use $\tilde{K} := (ab)^2(bc)(cd)^2a_xd_x$ instead of $[\tilde{H}, \tilde{H}]_1$ and $\tilde{I}' := (ab)^2(bc)(cd)^2(da)$ instead of $[\tilde{H}, \tilde{H}]_2$.

Remarks. (a) It follows from 5.5 Lemma C that the difference $\tilde{K} - [\tilde{H}, \tilde{H}]_1$ is a sum of covariants of type $\tilde{I}\tilde{H}$. This implies that $\tilde{K} \in J^2$ because $[H, H]_1 \in J^2$.

Similarly, we see that $\tilde{I}' = (ab)^2(bc)(cd)^2(da)$ belongs to J^2. In fact, $\tilde{I}' - [\tilde{H}, \tilde{H}]_2$ is a sum of invariants of type $\tilde{I}\tilde{I}$, again by 5.5 Lemma C.

(b) It follows from Proposition 11.3 and Remark 11.4(a) that the subspace corresponding to the covariant $[\tilde{H}, \tilde{H}]_1$ lies in the image of $\bigwedge^2(S^2M^*) \otimes V_2$ in $\mathcal{O}(W)_4$. This space is irreducible because $\bigwedge^2(S^2M^*) = S^{3,1}M^*$. Thus, in the notation of 10.2, we have $T_{[\tilde{H},\tilde{H}]_1} \otimes V_2 = S^{3,1} \otimes V_2$. Similarly, we see that $T_{[\tilde{H},\tilde{H}]_2} \subset S^2(S^2M^*) = S^4M^* \oplus S^{2,2}M^*$ and $T_{[[\tilde{H},\tilde{H}]_1,\tilde{H}]_2} \subset \bigwedge^3(S^3M^*) = S^{4,1,1}M^* \oplus S^{3,3}M^*$.

(c) Using Proposition 10.3 one can show that the covariants of type $\tilde{K}$ form the subspace $T_{\tilde{K}} \otimes V_2 = (S^{3,1} \oplus S^{2,1,1}) \otimes V_2 \subset \mathcal{O}(W)_4$ and that the remaining part of the isotypic component of V_2 in degree 4 is obtained from the product $\tilde{I}\tilde{H} = (ab)^3(cd)^2c_xd_x$ whose corresponding subspace is $T_{\tilde{I}\tilde{H}} \otimes V_2 = (S^{3,1} \oplus S^{2,1,1}) \otimes V_2$.

From now on we are mostly dealing with covariants of $\mathcal{O}(W)$ and we will remove the tilde from the notation above if there is no danger of confusion. From our considerations so far we obtain the following list of generators for the covariants of binary cubics:

$$\begin{gathered} F = a_x^3, \quad \Delta = (ab)a_x^2b_x^2, \quad H = (ab)^2a_xb_x, \quad K = (ab)^2(bc)(cd)^2a_xd_x, \\ I = (ab)^3, \quad I' = [H,H]_2, \quad I'' = [[H,H]_1,H]_2 \\ [F,H]_1, \quad [F,H]_2, \quad [F,H_1H_2]_3, \quad [F_1F_2,H_1H_2H_3]_6, \\ [F,K]_1, \quad [F,K]_2, \quad [F,HK]_3, \quad [F_1F_2,H_1H_2K]_6, \\ [\Delta,H]_1, \quad [\Delta,H]_2, \quad [\Delta,H_1H_2]_3, \quad [\Delta,H_1H_2]_4, \\ [\Delta,K]_1, \quad [\Delta,K]_2, \quad [\Delta,HK]_3, \quad [\Delta,HK]_4. \end{gathered} \tag{3}$$

12.4 Reduction by universal relations. We apply the results of the following section 13 to reduce further the list of generators. We check for every transvection listed above in 12.3(3) if there is suitable contraction such that Proposition 13.1 or 13.2 applies. In this case, the contraction is reducible and thus can be deleted from the list.

It is easy to see that Proposition 13.1 applies to the following transvections:

$$\begin{aligned} [\Delta,H]_1 : &\quad (ab)(bc)(cd)^2a_x^2b_xd_x \\ [\Delta,K]_1 : &\quad (ab)(bc)(cd)^2(de)(ef)^2a_x^2b_xf_x \\ [\Delta,H_1H_2]_3 : &\quad (ab)^2(bc)(cd)(de)(df)(ef)^2a_xc_x \\ [\Delta,HK]_3 : &\quad (ab)^2(bc)(cd)^2(de)(ef)(fg)(fh)(gh)^2a_xe_x \\ [F,K]_1 : &\quad (ab)(bc)^2(cd)(de)^2a_x^2e_x \\ [F,HK]_3 : &\quad (ab)^2(bc)(cd)^2(de)(ef)(eg)(fg)^2a_x \\ [F_1F_2,H_1H_2H_3]_6 : &\quad (ab)^2(ac)(bc)(cd)(de)^2(ef)(fg)(fh)(gh)^2 \\ [F_1F_2,H_1H_2K]_6 : &\quad (ab)^2(ac)(bc)(cd)(de)^2(ef)(fg)^2(gh)(hl)(hm)(lm)^2 \end{aligned}$$

In the same way we can use Proposition 13.2 to show that $[\Delta, K]_2$ is reducible:

$$[\Delta, K]_2 : \qquad (ab)(bc)(cd)^2(de)(ef)^2(fa)a_x b_x$$

12.5 Reduction by the lifting procedure. Let $[A, B]_\ell$ be one of the transvections of Lemma 12.2 where A is a covariant of $\mathcal{O}(W)/J$ and B a covariant of $\mathcal{O}(W_1)$. Recall that "lifting" to $\mathcal{O}(W)$ means to take first the image under the canonical homomorphism $\Phi\colon \mathcal{O}(W)/J \otimes \mathcal{O}(W_1) \to \bigoplus_i J^i/J^{i+1} = \operatorname{gr}_J \mathcal{O}(W)$ and then to lift it to $\mathcal{O}(W)$. Such a lift for $[A, B]_\ell$ is obtained by lifting A, B to $\tilde{A}, \tilde{B}$ and then forming the transvection $[\tilde{A}, \tilde{B}]_\ell$ in $\mathcal{O}(W)$.

Now assume that $B \in \mathcal{O}(W_1)_k$ and so $\Phi([A, B]_\ell) \in J^k/J^{k+1}$. If there is an ℓ-contraction T of $\tilde{A}$ and $\tilde{B}$ such that $T \in J^{k+1}$ then $\tilde{C} := [\tilde{A}, \tilde{B}]_\ell - T$ is also a lift of $[A, B]_\ell$. But according to 5.5 Lemma C this difference is expressible by lower transvections. Thus we can remove $[\tilde{A}, \tilde{B}]_\ell$ from our list of generators.

We claim that this method applies to the transvections

$$[\Delta, H]_2, \quad [\Delta, H_1 H_2]_4, \quad \text{and} \quad [\Delta, HK]_4.$$

(a) We have $H \in \mathcal{O}(W_1)_1$, and so $\Phi([\Delta, H]_2) \in J/J^2$. Consider the 2-contraction $(ab)^2(bc)(cd)(da)c_x d_x$ of $\tilde{\Delta}$ and $\tilde{H}$. It can be written in the form

$$\begin{aligned} 2(ab)^2(bc)(cd)(da)c_x d_x &= \frac{1}{2}(ab)\left((ab)^2(cd)^2 + (bc)^2(ad)^2 - (ac)^2(bd)^2\right) c_x d_x \\ &= (ab)^3(cd)^2 c_x d_x + (da)^2(ab)(bc)^2 d_x c_x - \\ &\qquad (ca)^2(ab)(bd)^2 c_x d_x \end{aligned}$$

(see Remark 11.4(b)). It shows that $(ab)^2(bc)(cd)(da)c_x d_x \in J^2$. In fact, the first expression is of type $\tilde{I}\tilde{H} \in J^2$ and the two others of type $\tilde{K}$ where $\tilde{K} \in J^2$ by Remark 12.3(a).

(b) We have $\Phi([\Delta, H_1 H_2]_4) \in J^2/J^3$. Consider the following 4-contraction of $\tilde{\Delta}$ and $\tilde{H}_1 \tilde{H}_2$:

$$C = (ab)^2(ac)(bd)(cd)(ce)(df)(ef)^2.$$

It can be considered as a 2-contraction of $A = (ab)^2(ac)(bd)(cd)c_x d_x$ with $\tilde{H} = (ef)^2 e_x f_x$. The first belongs to J^2 by (a) and the second to J. Thus $[A, \tilde{H}]_2 \in J^3$. From 5.5 Lemma C we see that $C - [A, \tilde{H}]_2$ is a sum of products of invariants of type $\tilde{I}' = (ab)^2(ac)(bd)(cd)^2$ and $\tilde{I} = (ef)^3$, and $\tilde{I}' \in J^2$ by Remark 12.3(a) and $\tilde{I} \in J$.

(c) We have $\Phi([\Delta, HK]_4) \in J^3/J^4$. Consider the following 4-contraction of $\tilde{\Delta}$ and $\tilde{H}\tilde{K}$:

$$D = (ab)^2(ac)(bd)(cd)(ce)(ef)^2(dg)(gh)^2(fh)$$

It can be considered as a 2-contraction of $A = (ab)^2(ac)(bd)(cd)c_x d_x$ with K where both belong to J^2 by (a) and Remark 12.3(a). As above, we find that $D - [A, K]_2$ is a sum of products $\tilde{I}'\tilde{I}' \in J^4$ and so $D \in J^4$.

Finally, we claim that only one of the two transvections $[F, K]_2$ and $[F, H_1H_2]_3$ is needed in the generating system. First, we can replace $[F, H_1H_2]_3$ by the 3-contractions $C = (ab)(ac)(ad)(bc)^2(de)^2e_x$ of F with H_1H_2. Now $C = [F, M]_2$ where $M := (ab)(bc)(bd)(cd)^2a_x^2$, because there is only one 2-contraction of F with M. Inspecting the list of generators given in 12.3(3) we see that the covariants of degree 4 and order 2 are linearly spanned by K, HI and $[\Delta, H]_2$. Since $[\Delta, H]_2$ is reducible (12.4), it follows that M is a linear combination of K and HI. Therefore, $C = [F, M]_2$ is a linear combination of $[F, K]_2$ and the reducible covariant $[F, H]_2I$ and the claim follows.

Summing up we have proved the following result. We write $C_{(d,e)}$ to indicate that the covariant is of degree d and order e. Moreover, we use the notation "$[A, B]_j : P$" in order to indicate that the symbolic expression P is a j-contraction of A and B and that P can be used in place of $[A, B]_j$ in the generating system. In some special cases, there is only one j-contraction and so $[A, B]_j = P$.

12.6 Proposition. *The following types form a generating system for the covariants of binary cubics V_3^m:*

$$F_{(1,3)} = a_x^3, \quad \Delta_{(2,4)} = (ab)a_x^2b_x^2, \quad H_{(2,2)} = (ab)^2a_xb_x, \quad I_{(2,0)} = (ab)^3,$$

$$K_{(4,2)} = [H, H]_1 : \ (ab)^2(bc)(cd)^2a_xd_x, \quad I_{(4,0)} = [H, H]_2 : \ (ab)^2(bc)(cd)^2(da),$$

$$I_{(6,0)} = [[H, H]_1, H]_2 : \ (ab)^2(bc)(cd)^2(de)(ef)^2(fa),$$

$$C_{(3,3)} = [F, H]_1 : \ (ab)(bc)^2a_x^2c_x, \quad C_{(3,1)} = [F, H]_2 = (ab)(ac)(bc)^2a_x$$

$$\text{One of the two types } D_{(5,1)} = [F, K]_2 = (ab)(bc)^2(cd)(de)^2(ea)a_x$$

$$\text{or } D'_{(5,1)} = [F, H^2]_3 : \ (ab)^2(bc)(cd)(ce)(de)^2a_x.$$

In particular, the covariants are generated by those of order ≤ 4 and degree ≤ 6.

We have already mentioned earlier that the subspace $T_C \otimes V_e$ generated by the covariants of type C (given as a symbolic expressions or as a transvection of those) is, in general, not an irreducible representation of $\mathrm{GL}(M) \times \mathrm{SL}_2$ (see 10.2, 10.3 and Remark 12.3(b)). We will now give the exact list of "irreducible" subspaces which minimally generate the covariants. This approach is beyond the methods employed by the classics and so the result seems to be new.

12.7 Theorem. *The covariants of binary cubics $V_3^m = M \otimes V_3$ are generated by 10 types of covariants corresponding to the following irreducible subspaces of*

$\mathcal{O}(M \otimes V_3)$:

(1) *Three types of invariants in degree 2, 4 and 6:*

$$\begin{aligned} \textstyle\bigwedge^2 M^* \otimes V_0 &= T_{(ab)^3} \otimes V_0 \subset \mathcal{O}(W)_2 \\ S^4 M^* \otimes V_0 &\subset T_{[H,H]_2} \otimes V_0 \subset \mathcal{O}(W)_4 \\ S^{3,3} M^* \otimes V_0 &\subset T_{[[H,H]_1,H]_2} \otimes V_0 \subset \mathcal{O}(W)_6 \end{aligned}$$

(2) *Two types of covariants of order 1 in degree 3 and 5:*

$$\begin{aligned} S^{2,1} M^* \otimes V_1 &\subset T_{[F,H]_2} \otimes V_1 \subset \mathcal{O}(W)_3 \\ S^{4,1} M^* \otimes V_1 &\subset T_{[F,K]_2} \otimes V_1 \subset \mathcal{O}(W)_5 \end{aligned}$$

(3) *Two types of covariants of order 2 in degree 2 and 4:*

$$\begin{aligned} S^2 M^* \otimes V_2 &\subset T_{H=(ab)^2 a_x b_x} \otimes V_2 \subset \mathcal{O}(W)_2 \\ S^{3,1} M^* \otimes V_2 &= T_{[H,H]_1} \otimes V_2 \subset T_{(ab)^2(bc)(cd)^2 a_x d_x} \otimes V_2 \subset \mathcal{O}(W)_4 \end{aligned}$$

(4) *Two types of covariants of order 3 in degree 1 and 3:*

$$\begin{aligned} M^* \otimes V_3 &= T_{F=a_x^3} \otimes V_3 \subset \mathcal{O}(W)_1 \\ S^3 M^* \otimes V_3 &\subset T_{[F,H]_1} \otimes V_3 \subset \mathcal{O}(W)_3 \end{aligned}$$

(5) *One type of covariant of order 4 in degree 2:*

$$\textstyle\bigwedge^2 M^* \otimes V_4 \subset T_{\Delta=(ab)a_x^2 b_x^2} \otimes V_4 \subset \mathcal{O}(W)_2$$

Proof. We first rewrite the types $C = C_{d,e}$ listed in Proposition 12.6 indicating what we know about the corresponding subspaces $T_C \otimes V_e \subset \mathcal{O}(W)_d$. For the covariants of type F, Δ, H and I they are known because in these cases the isotypic component of V_e in $\mathcal{O}(W)_d$ is irreducible (see the decompositions (1) at the beginning of §12). For $[H,H]_1$, $[H,H]_2$ and $[[H,H]_1,H]_2$ the description follows from Remark 12.3(b). For the other types we use the obvious fact that $T_{[A,B]_j} \subset T_A \otimes T_B$ for any $j \geq 0$. For the covariant $[F,H]_2$ of order 1 this gives $T_{[F,H]_2} \subset T_F \otimes T_H = M^* \otimes S^2 M^* = S^3 M^* \oplus S^{2,1} M^*$. But $S^3 M^* \otimes V_1$ does not occur in $\mathcal{O}(W)_3$ and so $T_{[F,H]_2} \otimes V_1 = S^{2,1} M^* \otimes V_1$ is irrducible.

$$
\begin{aligned}
F = a_x^3 : \quad & T_F \otimes V_3 = M^* \otimes V_3 \subset \mathcal{O}(W)_1 \\
\Delta = (ab) a_x^2 b_x^2 : \quad & T_\Delta \otimes V_4 = \textstyle\bigwedge^2 M^* \otimes V_4 \subset \mathcal{O}(W)_2 \\
H = (ab)^2 a_x b_x : \quad & T_H \otimes V_2 = S^2 M^* \otimes V_2 \subset \mathcal{O}(W)_2 \\
I = (ab)^3 : \quad & T_I \otimes V_0 = \textstyle\bigwedge^2 M^* \otimes V_0 \subset \mathcal{O}(W)_2 \\
[H,H]_1 : \quad & T_{[H,H]_1} \otimes V_2 = \textstyle\bigwedge^2 (S^2 M^*) \otimes V_2 = \\
& \qquad = S^{3,1} M^* \otimes V_2 \subset \mathcal{O}(W)_4 \\
[H,H]_2 : \quad & T_{[H,H]_2} \otimes V_0 \subset S^2(S^2 M^*) \otimes V_0 = \\
& \qquad = (S^4 M^* \oplus S^{2,2} M^*) \otimes V_0 \subset \mathcal{O}(W)_4 \\
[[H,H]_1, H]_2 : \quad & T_{[[H,H]_1,H]_2} \otimes V_0 \subset \textstyle\bigwedge^3 (S^2 M^*) \otimes V_0 = \\
& \qquad = (S^{4,1,1} M^* \oplus S^{3,3} M^*) \otimes V_0 \subset \mathcal{O}(W)_6 \\
[F,H]_1 : \quad & T_{[F,H]_1} \otimes V_3 \subset (M^* \otimes S^2 M^*) \otimes V_3 = \\
& \qquad = (S^3 M^* \oplus S^{2,1} M^*) \otimes V_3 \subset \mathcal{O}(W)_3 \\
[F,H]_2 : \quad & T_{[F,H]_2} \otimes V_1 = S^{2,1} M^* \otimes V_1 \subset \mathcal{O}(W)_3 \\
[F,K]_2 : \quad & T_{[F,K]_2} \otimes V_1 \subset (M^* \otimes S^{3,1} M^*) \otimes V_1 = \\
& \qquad = (S^{4,1} M^* \oplus S^{3,2} M^* \oplus S^{3,1,1} M^*) \otimes V_1 \subset \mathcal{O}(W)_5 \\
[F,H^2]_2 : \quad & T_{[F,H_1 H_2]_2} \otimes V_1 \subset (M^* \otimes S^2(S^2 M^*) \otimes V_1 = \\
& \qquad = (S^5 M^* \oplus S^{4,1} M^* \oplus S^{3,2} M^* \oplus S^{2,2,1} M^*) \otimes V_1 \\
& \qquad\qquad \subset \mathcal{O}(W)_5
\end{aligned}
$$

In order to reduce this list further, we proceed by degree and try to determine the subspaces which are obtained by forming products of previous covariants. For this we can use the following remark. Let A and B be two covariants with corresponding irreducible subspaces $T_A \otimes V_e$ and $T_B \otimes V_f$, and denote by $T \subset T_A \otimes T_B$ the CARTAN component. Then we get for the subspace corresponding to the product AB the following inclusions

$$
T \otimes V_{e+f} \subset T_{AB} \otimes V_{e+f} \subset (T_A \otimes T_B) \otimes V_{e+f}. \tag{$*$}
$$

(a) In degree 1 there is only the covariant F and $T_F = M^*$ is irreducible.

(b) In degree 2 we have three types, Δ, H and I, where $T_\Delta = \bigwedge^2 M^*$, $T_H = S^2 M^*$ and $T_I = \bigwedge^2 M^*$ and all these spaces are irreducible.

(c) In degree 3 we have two types, $[H,F]_1$ and $[H,F]_2$. We have already seen that $T_{[H,F]_2} \otimes V_1 = S^{2,1} \otimes V_1$ is irreducible. Next we observe that $S^{2,1} M^* \otimes V_1$ and

$S^3M^*\otimes V_3$ cannot be obtained from lower degree covariants according to $(*)$. In fact, for all covariants C of degree 2 the order of $F\cdot C$ is ≥ 3, and $T_F\otimes T_I = M^*\otimes\bigwedge^2 M^* = S^{2,1}M^*\oplus\bigwedge^3 M^*$. On the other hand, again by $(*)$, $T_{F\cdot I}\supset S^{2,1}M^*$. Hence, we can replace the type $[F,H]_1$ by the irreducible subspace $S^3\otimes V_3\subset T_{[F,H]_1}\otimes V_3$ which has to occur in the minimal generating system in degree 3.

(d) In degree 4 we have again two types, $[H,H]_1$ and $[H,H]_2$. We have already seen that $T_{[H,H]_1} = S^{3,1}M^*$ is irreducible and that $T_{[H,H]_2}\subset S^4M^*\oplus S^{2,2}M^*$. By $(*)$ it follows that $\bigwedge^2 M^*\otimes\bigwedge^2 M^*\supset T_{I\cdot I}\supset S^{2,2}M^*$ which shows that we can replace $[H,H]_2$ by $S^4M^*\otimes V_0$.

(e) In degree 5 we have only to deal with $[F,K]_2$ where $T_{[F,K]_2}\subset S^{4,1}M^*\oplus S^{3,2}M^*\oplus S^{3,1,1}M^*$. First we observe that $T_{[F,H]_2\cdot I}\supset S^{3,2}M^*$ by $(*)$. Then we know that we could replace $[F,K]_2$ by $[F,H^2]_2$ and that $S^{3,1,1}M^*$ does not occur in $T_{[F,H^2]_2}$. Thus we can replace $[F,K]_2$ by $S^{4,1}M^*\otimes V_1$.

(f) We are left with the invariant $[[H,H]_1,H]_2$ in degree 6 where $T_{[[H,H]_1,H]_2}\subset S^{4,1,1}M^*\oplus S^{3,3}M^*$. If we denote by I_4 the invariant corresponding to the subspace $S^4M^*\otimes V_0$ (see (d)) we obtain from $(*)$ that $S^{4,1,1}M^*\subset T_{I\cdot I_4}$. Moreover, $T_{I^3}\supset S^{3,3}M^*$. But $S^{3,3}M^*\otimes V_0$ has multiplicity 2 in $\mathcal{O}(W)_6$. Therefore, we have to add $S^{3,3}M^*\otimes V_0$ to obtain a minimal generating system.

This completes the proof of the Theorem. □

§13. Two universal relations for symbolic expressions

Recall that a symbolic expression is said to be *reducible* if it can be written as a linear combination (with rational coefficients) of decomposable symbolic expressions. In the following we describe two general types of reducible symbolic expressions. This has been used in the determination of the generating set for the covariants of cubics in the previous paragraph. All the relations below are relations in the symbolic algebra $\mathrm{Symb}_{\mathcal{J}}$ (3.4).

The starting point is the following classical relation which shows that the symbolic expression $(ab)(bc)a_x^{p_a-1}b_x^{p_b-2}c_x^{p_c-1}$ is reducible:

$$2(ab)(bc)a_xc_x = -(ab)^2c_x^2-(bc)^2a_x^2+(ca)^2b_x^2$$

"Polarizing" this relation we find an equation in 6 symbols a_1,a_2,b_1,b_2,c_1,c_2, each of order 1 which can also be considered as a relation in $\underbrace{V_1\otimes V_1\otimes\cdots\otimes V_1}_{6\text{ times}}$:

$$\begin{aligned}\frac{1}{2}\sum_{\sigma,\tau,\rho\in\mathcal{S}_2}(a_{\sigma1}b_{\tau1})(b_{\tau2}c_{\rho1})a_{\sigma2x}c_{\rho2x} &= (a_1b_1)(a_2b_2)c_{1x}c_{2x}+(a_1b_2)(a_2b_1)c_{1x}c_{2x}\\ &\quad+(b_1c_1)(b_2c_2)a_{1x}a_{2x}+(b_1c_2)(b_2c_1)a_{1x}a_{2x}\\ &\quad+(a_1c_1)(a_2c_2)b_{1x}b_{2x}+(a_1c_2)(a_2c_1)b_{1x}b_{2x}\end{aligned}$$

(This formula and all following ones can easily be checked by using a Computer Algebra Program like Maple.) At the left hand side we can use sucessively relations of the form

$$(a_1b_1)(b_2c_2)a_{2x}c_{1x}+(a_1b_1)(b_2c_1)a_{2x}c_{2x} = 2(a_1b_1)(b_2c_2)a_{2x}c_{1x}+(a_1b_1)(c_2c_1)a_{2x}b_{2x}$$

to end up with the following relation:

$$\begin{aligned} 4(ab)(b'c')a'_xc_x &= 2(ab)(cc')a'_xb'_x + 2(aa')(bc)b'_xc'_x \\ &\quad - 2(ac)(bb')a'_xc'_x + (aa')(cc')b_xb'_x - (bb')(cc')a_xa'_x - (aa')(bb')c_xc'_x \\ &\quad - (ab)(a'b')c_xc'_x - (ab')(a'b)c_xc'_x - (bc)(b'c')a_xa'_x - (bc')(b'c)a_xa'_x \\ &\quad + (ac)(a'c')b_xb'_x + (ac')(a'c)b_xb'_x \qquad (*) \end{aligned}$$

Each term on the right hand side can be written as a product of two factors where one contains only one letter and the other the two others, e.g. $(ab)(cc')a'_xb'_x = (ab)a'_xb'_x{\cdot}(cc')$ or $(bb')(cc')a_xa'_x = (bb'){\cdot}(cc'){\cdot}a_xa'_x$ or $(bc')(b'c)a_xa'_x = (bc)(b'c'){\cdot}a_xa'_x$.

13.1 Proposition. *Let P, Q, R be three disjoint symbolic expressions of order ≥ 2 and let T be the symbolic expression obtained from the product PQR by choosing a 1-contraction of P and Q and a 1-contraction of Q and R. Then T is reducible.*

Proof. We can write $P = \bar{P}a_xa'_x$, $Q = \bar{Q}b_xb'_x$ and $R = \bar{R}c_xc'_x$ such that $T = \bar{P}\bar{Q}\bar{R}(ab)(b'c')a'_xc_x$, i.e. we contract a_xb_x to (ab) and $b'_xc'_x$ to $(b'c')$.

Multiplying the relation $(*)$ above with $\bar{P}\bar{Q}\bar{R}$ we find

$$4T = 2\bar{P}\bar{Q}\bar{R}(ab)(cc')a'_xb'_x + 2\bar{P}\bar{Q}\bar{R}(aa')(bc)b'_xc'_x - 2\bar{P}\bar{Q}\bar{R}(ac)(bb')a'_xc'_x + \cdots$$

It is easy to see that all terms on the right hand side are decomposable into products of 2 (or 3) factors, e.g. $\bar{P}\bar{Q}\bar{R}(ab)(cc')a'_xb'_x = \bar{P}\bar{Q}(ab)a'_xb'_x \cdot \bar{R}(cc')$. Hence the claim follows. □

We can proceed in the same way, starting with the classical relation

$$2(ab)(bc)(cd)(da) = (ab)^2(cd)^2 + (ad)^2(bc)^2 - (ac)^2(bd)^2$$

First we get

$$\begin{aligned} \frac{1}{2}\sum_{\sigma,\tau,\rho,\mu\in\mathcal{S}_2} (a_{\sigma1}b_{\tau1})(b_{\tau2}d_{\mu1})(a_{\sigma2}c_{\rho1})(c_{\rho2}d_{\mu2}) = \\ (a_1b_1)(a_2b_2)(c_1d_1)(c_2d_2) + (a_1b_1)(a_2b_2)(c_1d_2)(c_2d_1) + \\ (a_1b_2)(a_2b_1)(c_1d_1)(c_2d_2) + (a_1b_2)(a_2b_1)(c_1d_2)(c_2d_1) + \\ (a_1c_1)(a_2c_2)(b_1d_1)(b_2d_2) + (a_1c_1)(a_2c_2)(b_1d_2)(b_2d_1) + \\ (a_1c_2)(a_2c_1)(b_1d_1)(b_2d_2) + (a_1c_2)(a_2c_1)(b_1d_2)(b_2d_1) \end{aligned}$$

Again, the left hand side can be sucessively rewritten using realtions of the form

$$(a_1b_1)(a_2c_1)(b_2d_1)(c_2d_2) + (a_1b_1)(a_2c_1)(b_2d_2)(c_2d_1) = \\ 2(a_1b_1)(a_2c_1)(b_2d_1)(c_2d_2) + (a_1b_1)(a_2c_1)(b_2c_2)(d_2d_1)$$

to obtain the relation

$$\begin{aligned}16(ab)(a'c)(b'd)(c'd') = -(&(ab)(a'c)(b'c')(d'd) + (a'b)(ac)(b'c')(d'd)+ \\ &(ab)(a'c')(b'c)(d'd) + (a'b)(ac')(b'c)(d'd)+ \\ &(ab')(a'c)(bc')(d'd) + (a'b')(ac)(bc')(d'd)+ \\ &(ab')(a'c')(bc)(d'd) + (a'b')(ac')(bc)(d'd)) \\ &- 2((a'a)(bc)(b'd)(c'd') + (a'a)(bc')(b'd)(cd')+ \\ &(a'a)(b'c)(bd)(c'd') + (a'a)(b'c')(bd)(cd')) \\ &- 4((ab)(a'd')(b'd)(cc') + (ab')(a'd')(bd)(cc')) \\ &- 8(ad)(a'c)(bb')(c'd') \\ &+ (ab)(a'b')(cd)(c'd') + (ab)(a'b')(cd')(c'd)+ \\ &(ab')(a'b)(cd)(c'd') + (ab')(a'b)(cd')(c'd)+ \\ &(ac)(a'c')(bd)(b'd') + (ac)(a'c')(bd')(b'd)+ \\ &(ac')(a'c)(bd)(b'd') + (ac')(a'c)(bd')(b'd) \qquad (**)\end{aligned}$$

Again, every term in the right hand side is decomposable into a product of two "disjoint" factors, i.e. one containing two letters and the other containing the remaining two letters. As a consequence we obtain the following result.

13.2 Proposition. *Let P, Q, R, S be four disjoint symbolic expressions of order at least 2. Define the symbolic expression T by forming the product $PQRS$ choosing a 1-contraction for each of the four pairs (P,Q), (Q,R), (R,S) and (S,P). Then T is reducible.*

Proof. We can write $P = \bar{P}a_xa'_x$, $Q = \bar{Q}b_xb'_x$, $R = \bar{R}c_xc'_x$, $S = \bar{S}d_xd'_x$ such that $T = \bar{P}\bar{Q}\bar{R}\bar{S}(ab)(b'c')(cd)(d'a')$. Hence, multiplying the above relation with $\bar{P}\bar{Q}\bar{R}\bar{S}$ we see that $16T$ is expressed as a sum of decomposable terms. □

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