ON A QUESTION OF YOSEF STEIN

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Abstract. Let $\varphi\colon \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial map whose jacobian is a constant $\neq 0$. Assume that φ is not an isomorphism. Then we show that the intersection of the iterated images ${\varphi^*}^i(\mathbb{C}[x,y])$ of the coordinate ring under the comorphism φ^* is the ring of constants $\mathbb{C}$. This answers a question asked by Yosef Stein.

During the Curaçao Conference on "Polynomial Automorphisms of Affine Space" Yosef Stein posed the following problem. Let $\varphi = (f,g)\colon\ \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial map ($f, g \in \mathbb{C}[x,y]$) and denote by $\sigma\colon \mathbb{C}[x,y] \to \mathbb{C}[x,y]$ the corresponding algebra homomorphism, i.e., $\sigma(x) = f$, $\sigma(y) = g$. Assume that

$$\operatorname{Jac}\varphi := \det \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = 1$$

and that φ is not an isomorphism.

Problem. What is the intersection of the images $\sigma^i(\mathbb{C}[x,y])$, $i = 1, 2, \dots$ *?*

The answer turns out to be the following:

Theorem. *Under the assumptions above we have*

$$\bigcap_{i=0}^{\infty} \sigma^i(\mathbb{C}[x,y]) = \mathbb{C}.$$

Let us point out that the assumption Jac $= 1$ is essential. In fact, let $f \in \mathbb{C}[x,y]$ be any polynomial which is non-linear in y. Then the morphism $\varphi(x,y) := (x, f(x,y))$ has the property that $\bigcap_i \sigma^i(\mathbb{C}[x,y]) = \mathbb{C}[x]$.

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The proof of the Theorem is based on a series of lemmas. We first remark that a birational morphism $\varphi\colon \mathbb{C}^2 \to \mathbb{C}^2$ with finite fibers is an isomorphism. This is well known; it follows easily from [Kra85, Lemma II.3.4 and Bemerkung].

Lemma 1. *Let $\sigma\colon \mathbb{C}[x,y] \to \mathbb{C}[x,y]$ be a algebra homomorphism which is not birational (i.e., which does not induce an isomorphism of the field of fractions) and let $R := \bigcap_{i=0}^{\infty} \sigma^i(\mathbb{C}[x,y])$. Then σ induces an isomorphism $\tilde{\sigma}\colon R \to R$. Moreover, R is integrally closed in $\mathbb{C}[x,y]$, and either $R = \mathbb{C}$ or R is a polynomial ring in one variable.*

Proof. Let $R_i := \sigma^i(\mathbb{C}[x,y])$ and $K_i := \operatorname{Quot} R_i$ its field of fractions. Then $[K_i : K_{i+1}] > 1$ and therefore $[\mathbb{C}(x,y) : \bigcap K_i] = \infty$. Since $K := \operatorname{Quot} R \subset \bigcap K_i$ we see that $\operatorname{tr}\deg_{\mathbb{C}} K \leq 1$. If $K = \mathbb{C}$ then $R = \mathbb{C}$ and we are done. Otherwise we have $K = \mathbb{C}(z)$ by Lüroth's theorem. Since every $R_i \simeq \mathbb{C}[x,y]$ is integrally closed in K_i, it follows that R is integrally closed in K, and since the only invertible elements in R are the constants we finally obtain that R is a polynomial ring in one variable.

It is clear that σ induces an automorphism $\tilde{\sigma}$ of R. Let $\tilde{R}$ be the integral closure of R in $\mathbb{C}[x,y]$. Then σ induces an isomorphism of $\tilde{R}$, too. It follows that for any $h \in \tilde{R}$ and every i there is an $h_i \in \mathbb{C}[x,y]$ such that $\sigma^i(h_i) = h$, i.e., $h \in R_i$ for all i. Thus $\tilde{R} = R$. □

From now on we assume that $R := \bigcap_{i=0}^{\infty} \sigma^i(\mathbb{C}[x,y]) = \mathbb{C}[p]$. In geometric terms this means that we have the following commutative diagram

$$\begin{tikzcd}
\mathbb{C}^2 \arrow[r, "\varphi"] \arrow[d, "p"'] & \mathbb{C}^2 \arrow[d, "p"] \\
\mathbb{C} \arrow[r, "\tilde{\varphi}"] & \mathbb{C}
\end{tikzcd} \tag{$*$}$$

where $\tilde{\varphi}$ is an isomorphism. In particular, there are three possibilities for $\tilde{\varphi}$:

(1) $\tilde{\varphi}$ is of finite order;
(2) $\tilde{\varphi}$ is a translation $z \mapsto z + a$;
(3) $\tilde{\varphi}$ has a unique fixed point and is not of finite order.

Another consequence of the diagram $(*)$ is the following. Denote by $F_a := p^{-1}(a)$ the fiber of p over $a \in \mathbb{C}$. Since R is integrally closed in $\mathbb{C}[x,y]$ the generic fiber of p is connected, and also smooth (by Sard's theorem). Moreover, $\varphi^{-1}(F_a) = F_b$ if $\tilde{\varphi}(b) = a$ and the induced morphism $\varphi|_{F_b}\colon F_b \to F_a$ is étale.

Lemma 2. *If $a \in \mathbb{C}$ is a fixed point under $\tilde{\varphi}$ then the fiber F_a is irreducible and reduced.*

Proof. We can assume that $a = 0$ and have to show that the polynomial p is irreducible. By definition, $\sigma(p) = \lambda p$ for some $\lambda \in \mathbb{C}$, $\lambda \neq 0$. Hence, for any prime factor q of p there is a $k > 0$ such that $\sigma^k(q) = \nu q$ for some non-zero $\nu \in \mathbb{C}$. Thus, $q \in \sigma^i(\mathbb{C}[x,y])$ for all i, and so $q \in R = \mathbb{C}[p]$ and $p = q$. □

In the following we will use some basic facts about the Euler characteristic of algebraic varieties. Our reference is the appendix of [KrP85].

Lemma 3. *Let C and D be two smooth affine curves of the same Euler characteristic $e := \chi(C) = \chi(D)$, and let $\rho: C \to D$ be a morphism of degree $d > 1$. Then one of the following two cases holds:*

(i) *$e = 1$, both curves C and D are isomorphic to $\mathbb{C}$, and ρ is a finite morphism;*
(ii) *$e = 0$, both curves C and D are isomorphic to $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and ρ is a finite étale covering.*

If, in addition, ρ is étale then only case (ii) *is possible.*

Proof. There is a dense open subset $U \subset D$ such that $\#\rho^{-1}(x) = d$ for all $x \in U$ and $\#\rho^{-1}(x) < d$ for all $x \in D \setminus U$. Let $k := \#D \setminus U$. Then

$$e = \chi(C) = \chi(\rho^{-1}(U)) + \chi(\rho^{-1}(D \setminus U)) = d(e - k) + m$$

where $m \leq k(d-1)$. It follows that

$$(d-1)e = dk - m \geq k \geq 0, \tag{$**$}$$

hence $e \geq 0$. It is well known that an affine smooth curve of Euler characteristic 1 is isomorphic to $\mathbb{C}$ and that one of Euler characteristic 0 is isomorphic to $\mathbb{C}^*$. It is also clear that every morphism $\rho: \mathbb{C} \to \mathbb{C}$ of degree > 1 is ramified. Moreover, it follows from $(**)$ that for $e = 0$ we have $k = 0$, i.e., ρ is a finite covering of degree d in this case. □

The following result is not needed in the proof of the theorem. We just state it to complete the picture.

Remark. *Let C be an irreducible affine curve with singularities and let $\rho: C \to C$ be an étale morphism. Then ρ is an isomorphism.*

Proof. Let $\eta: \tilde{C} \to C$ be the normalization and consider the following pull-back diagram:

$$\begin{tikzcd}
Y \arrow[r, "\rho'"] \arrow[d, "\eta'"'] & \tilde{C} \arrow[d, "\eta"] \\
C \arrow[r, "\rho"] & C
\end{tikzcd}$$

We claim that $Y \simeq \tilde{C}$ and that η' is normalization. In fact, ρ' is étale, hence Y smooth, and η' is finite and birational and therefore the normalization of C. It follows from Lemma 2 that ρ' is a finite covering, hence $\eta \circ \rho' = \rho \circ \eta'$ is finite and therefore ρ is finite, too. If $x \in C$ is a singular point, then $\#\rho^{-1}(x) = d = \deg \rho$, and all points in $\rho^{-1}(x)$ are singular. This is possible only for $d = 1$. □

Now we are ready to give the proof of the main theorem.

Proof of the Theorem. By assumption the morphism $\varphi: \mathbb{C}^2 \to \mathbb{C}^2$ is étale of degree $d > 1$. Let us also assume that $R = \bigcap \sigma^i(\mathbb{C}[x, y]) \neq \mathbb{C}$. We want to show that this leads to a contradiction. We use the notation introduced above. In particular, we have the following diagram:

$$\begin{tikzcd}
\mathbb{C}^2 \arrow[r, "\varphi"] \arrow[d, "p"'] & \mathbb{C}^2 \arrow[d, "p"] \\
\mathbb{C} \arrow[r, "\tilde{\varphi}", "\simeq"'] & \mathbb{C}
\end{tikzcd}$$

Case 1: $\tilde{\varphi}$ is of finite order. We can replace $\tilde{\varphi}$ by a suitable power and assume without loss of generality that $\tilde{\varphi} = \mathrm{Id}$. It then follows from Lemma 2 and 3 that every fiber is irreducible and reduced and that the generic fiber is smooth and isomorphic to $\mathbb{C}^*$, i.e., there is an open subset $U \subset \mathbb{C}$ such that $F_a \simeq \mathbb{C}^*$ for all $a \in U$. Set $\{a_1, \ldots, a_r\} := \mathbb{C} \setminus U$. Since all fibers over U have Euler characteristic 0 we get

$$1 = \chi(\mathbb{C}^2) = \chi(F_{a_1}) + \cdots + \chi(F_{a_r}).$$

In particular, at least one of the fibers F_{a_i} has Euler characteristic equal to 1. (Recall that for any affine curve the Euler characteristic is ≤ 1.) By a fundamental result due to Zaidenberg and Lin [ZaL83] such a curve is equivalent to one given by an equation of the form $x^r + y^s = 0$ where $r,\ s \geq 1$ are coprime integers. This implies that after a suitable polynomial change of coordinates in $\mathbb{C}^2$ we may assume that $p(x, y) = x^r + y^s$. Therefore, the fiber F_a is given by the equation $x^r + y^s = a$, and an easy calculation shows that $\chi(F_a) = 1 - (r - 1)(s - 1)$ for $a \neq 0$. This leads to a contradiction because this number is always different from zero for coprime r, s.

Case 2: $\tilde{\varphi}$ is a translation. It is well known that there is an non-empty open subset $U \subset \mathbb{C}$ with the property that all fibers over U are smooth and have the same Euler characteristic. Since $\tilde{\varphi}(U) \cap U$ is open and non-empty we can apply Lemma 3 and deduce that the generic fiber is isomorphic to $\mathbb{C}^*$. Shrinking U we can even assume that all fibers over U are isomorphic to $\mathbb{C}^*$. Now let $F = F_b$ be an arbitrary fiber. There is clearly an integer $m \geq 0$ such that $a := \sigma^m(b) \in U$. It follows that we have an étale morphism $F \to F_a \simeq \mathbb{C}^*$ which implies that F is smooth and that every irreducible component has Euler characteristic ≤ 0. (There is no non-constant morphism $\mathbb{C} \to \mathbb{C}^*$.) Thus, $\chi(F) \leq 0$ for every fiber of p and $= 0$ for the generic fiber which leads to a contradiction.

Case 3: $\tilde{\varphi}$ has a unique fixed point and is not of finite order. Here we use a combination of the arguments above. Case 2 can be applied to the open set $\dot{\mathbb{C}} := \mathbb{C} \setminus \{\text{fixed point}\}$ and implies that the generic fiber has Euler characteristic 0 and that all other fibers over $\dot{\mathbb{C}}$ have Euler characteristic ≤ 0. Thus the remaining fiber F_0 over the fixed point must have Euler characteristic 1, and F_0 is irreducible and reduced by Lemma 2. The same argument as in case 1 using again [ZaL83] leads to a contradiction. □

References

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