A multihomogenization technique for the study of essential dimension of algebraic groups

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Outline

Essential dimension of algebraic groups

- k an arbitrary field
- G a (smooth linear) algebraic group over k (often finite)

Definition (of $\operatorname{ed}_k G$)

The essential dimension of G is the least transcendence degree over k of a field of definition of a generic G-torsor $T \to \operatorname{spec} K$ (where K/k is some field extension).

It is the essential dimension of the Galois cohomology functor $H^1(-, G)\colon \operatorname{Fields}_k \to \operatorname{Sets}$.

Covariants $\varphi\colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(W)$

Definition

A covariant of G is a G-equivariant rational map

$$\varphi\colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(W)$$

where V, W are G-modules.

- φ is called generically free if $\overline{\varphi(\mathbb{A}(V))}$ is generically free.
- $$\dim \varphi := \text{ dimension of } \overline{\varphi(\mathbb{A}(V))}.$$

Essential dimension in terms of covariants

$\operatorname{ed}_k G = \min\{\dim \varphi \mid \varphi \text{ gen. free covariant of } G \text{ over } k\} - \dim G.$

For G-modules V, W with $\mathbb{A}(V)$ and $\mathbb{A}(W)$ gen. free there exists $\varphi\colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(W)$ gen. free such that $\dim \varphi - \dim G = \operatorname{ed}_k G$.

Motivation for using covariants $\varphi\colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(W)$

- Computational: explicit description, search in low degrees

$G = S_n$ and P the canonical n-dim. permutation module

Every covariant $\varphi\colon \mathbb{A}(P) \dashrightarrow \mathbb{A}(P)$ is of the form

$$\varphi = \frac{1}{s} \cdot (f_1, f_2, \ldots, f_n)$$

where $s \in k[P]^{S_n}$, $f_1 \in k[P]^{\{1\} \times S_{n-1}}$ and $f_i = f_1^{(1i)}$ for $i = 2, \ldots, n$.

- Relation with classical results
- Flexibility by modifying covariants and deforming $\varphi(\mathbb{A}(V)) \subseteq \mathbb{A}(W)$:
  - $\varphi \rightsquigarrow \tilde{\varphi} = f \cdot \varphi$ where $f \in k(V)^G$
  - (multi-)homogenization

Goal:

Find generically free covariants $\varphi\colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(W)$ of minimal dimension with useful special properties.

- Regular covariants: $\varphi = (f_1, f_2, \ldots, f_n)$ with all $f_i \in k[V]$ polynomials.
- Homogeneous covariants: All f_i are homogeneous of the same degree $m \in \mathbb{Z}$ (possibly some $f_i = 0$)
  Equivalently: $\varphi(\lambda v) = \lambda^m \varphi(v)$ for an indeterminate λ.

Why is homogeneity useful?

- One reason: Let $p \neq \operatorname{char} k$ be a prime with $p \nmid m, p \nmid |Z(G)|$. Then φ can be turned into a covariant of $G \times \mathbb{Z}/p\mathbb{Z}$.
- Another reason: If $m \neq 0$ then $\overline{\varphi(\mathbb{A}(V))}$ is a cone (i.e. stable under the action $\mathbb{G}_m \times \mathbb{A}(W) \to \mathbb{A}(W), (\lambda, x) \mapsto \lambda x$).

Question:

Does every algebraic group G have a generically free homogeneous covariant $\varphi\colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(W)$ of dimension $\mathrm{ed}_k G + \dim G$?

Answer:

No!

But: (Kraft and Schwarz)

If W is irreducible and G is finite it works! Replace φ by its highest degree component $\varphi_{\max}\colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(W)$.

- $\varphi_{\max}$ is again a covariant.
- $\varphi_{\max}$ is faithful.
- $\dim \varphi_{\max} \leq \dim \varphi$.

Problem:

Few groups admit an irreducible generically free representation!

Outline

Multihomogeneous rational maps and covariants

- $V = \bigoplus_{i=1}^{m} V_i$
- $W = \bigoplus_{j=1}^{n} W_j$

- A rational map (covariant)

$$\varphi = (\varphi_1, \dots, \varphi_n) \colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(W)$$

is called multihomogeneous (with respect to the chosen decompositions of V and W) if for some $m_{ij} \in \mathbb{Z}$

$$\varphi_j(v_1, \dots, \lambda v_i, \dots v_n) = \lambda^{m_{ij}} \varphi_j(v_1, \dots, v_n).$$

- The degree matrix $M_\varphi := (m_{ij}) \in \mathrm{Mat}_{m \times n}(\mathbb{Z})$ gives useful information! Assume $\varphi_j \neq 0$ for all j in the sequel.

A simple example

- $G = (\mathbb{Z}/p\mathbb{Z})^n$.
- k a field containing a primitive pth root of unity.
- V canonical n-dim. faithful G-module.
- $\varphi\colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(V)$ multihomogeneous (wrt. irred. decomp.)
- Then the degree matrix has the form

$$M_\varphi = \begin{pmatrix} 1 + a_{11}p & a_{12}p & \dots & \dots & a_{1n}p \\ a_{21}p & 1 + a_{22}p & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ a_{n1}p & a_{n2}p & \dots & \dots & 1 + a_{nn}p \end{pmatrix}$$

- In general: M_φ is influenced by central elements of G.

Multihomogenization

arbitrary covariant $\rightsquigarrow$ multihomogeneous covariant

Multihomogenization is a procedure to get from arbitrary to multihomogeneous covariants while preserving good properties.

- Let $V = \bigoplus_{i=1}^{m} V_i$ be a decomposition into G-submodules.
- $T_V := (\mathbb{G}_m)^m \hookrightarrow \mathrm{GL}(V)$ (similarly for $W = \bigoplus_{j=1}^{n} W_j$)
- Let $\lambda\colon \mathbb{G}_m \to T_V$ and $\mu\colon \mathbb{G}_m \to T_W$ 1-parameter subgroups and $\varphi\colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(W)$ a covariant.
- Consider (for $t \in \mathbb{G}_m$)

$$\varphi_{\lambda,\mu}^{(t)}\colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(W), \quad v \mapsto \mu(t)\varphi\big(\lambda(t)v\big)$$

- Each $\varphi_{\lambda,\mu}^{(t)}$ is a covariant and their images are all isomorphic.
- For well chosen λ, μ the family $\varphi_{\lambda,\mu}^{(t)}$ has a limit in $t = 0$.

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- For well chosen λ, μ the family $\varphi_{\lambda,\mu}^{(t)}$ has a limit in $t = 0$.
- For given λ there exists unique μ such that $\varphi_{\lambda,\mu}^{(0)}$ is defined and its components in $W_1, \ldots, W_n$ are $\neq 0$. For such μ define $H_\lambda(\varphi) := \varphi_{\lambda,\mu}^{(0)}$.

Illustration of multihomogenization

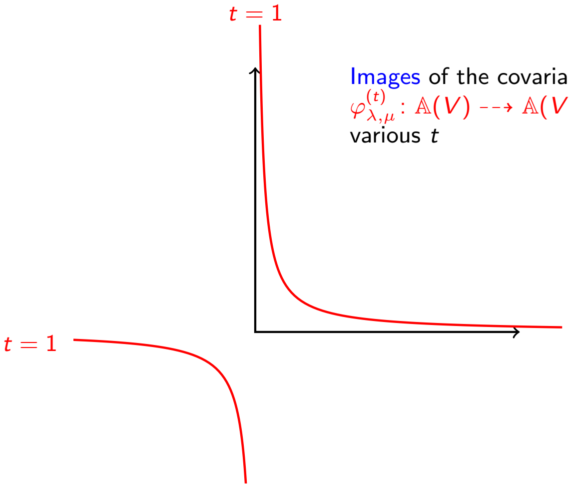
- V the 2-dim. permutation $G = S_2$-module over $k = \mathbb{R}$
- $$\varphi\colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(V), \quad (x, y) \mapsto \left(\frac{x}{y}, \frac{y}{x}\right)$$

  will be multihomogenized wrt. $V = k(1,1) \oplus k(1,-1)$.
- Choose $\lambda\colon \mathbb{G}_m \to \mathbb{G}_m^2, t \mapsto (1, t), \quad \mu(t) = (1, t^{-1})$.
- In basis $(1,1), (1,-1)$ (and coordinates a, b)

$$\mu(t)\varphi(\lambda(t)(a,b)) = \left(1 \cdot \frac{a^2 + (tb)^2}{a^2 - (tb)^2}, t^{-1} \cdot \frac{2a(bt)}{a^2 - (bt)^2}\right)$$
$$= \left(\frac{a^2 + t^2 b^2}{a^2 - t^2 b^2}, \frac{2ab}{a^2 - b^2 t^2}\right) \xrightarrow{t \to 0} \left(1, 2\frac{b}{a}\right)$$

- $H_\lambda(\varphi)(x, y) = \left(1 + 2\frac{x-y}{x+y}, 1 - 2\frac{x-y}{x+y}\right)$ has degree $\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$.

Illustration of multihomogenization



Images of the covariants $\varphi^{(t)}_{\lambda,\mu} \colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(V)$ for various t

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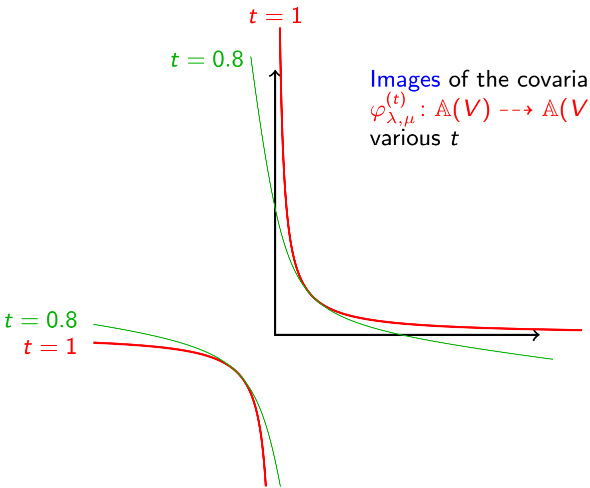

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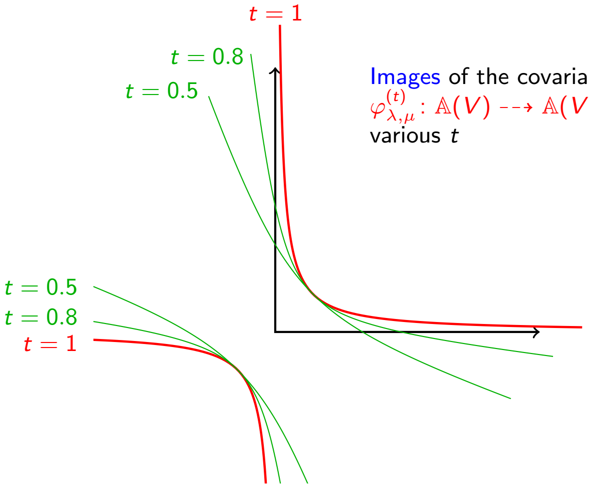

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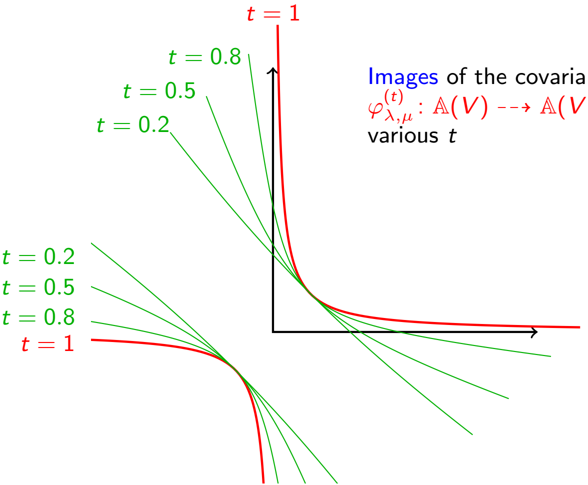

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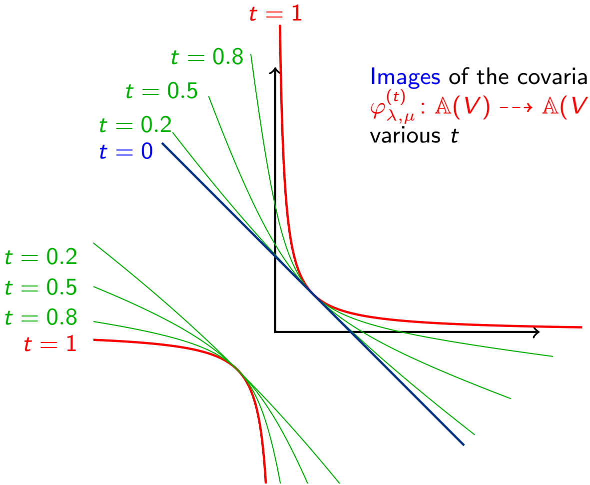

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Multihomogenization continued

- $$\varphi^{(t)}_{\lambda,\mu} \colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(W), \quad v \mapsto \mu(t)\varphi(\lambda(t)v)$$
- For given λ there exists unique μ such that $\varphi^{(0)}_{\lambda,\mu}$ is defined and its components in $W_1, \ldots, W_n$ are $\neq 0$.
- For such μ define $H_\lambda(\varphi) := \varphi^{(0)}_{\lambda,\mu}$.

Theorem

- *$H_\lambda(\varphi)$ is a covariant.*
- $\dim H_\lambda(\varphi) \leq \dim \varphi$.
- *For suitable choice of λ the covariant $H_\lambda(\varphi)$ is multihomogeneous.*
- *$W_1, \ldots, W_n$ irreducible $\Rightarrow$ G acts faithfully on $H_\lambda(\varphi)(\mathbb{A}(V))$.*

Corollary

Assume that G is finite and admits a completely reducible faithful representation $W = \bigoplus_{j=1}^{n} W_j$. Then (for any $V = \bigoplus_{i=1}^{m} V_i$)

$$\operatorname{ed}_k G = \min\{\dim \varphi \mid \varphi\colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(W) \text{ faithful multihomogeneous covariant}\}.$$

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Question:

Why are multihomogeneous covariants useful?

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Theorem about M_φ: Assume G finite, constant, V_i, W_j irreducible

Let $Z(G,k)$ denote the subgroup $\{g \in Z(G) \mid \zeta_{\operatorname{ord} g} \in k\} \subseteq Z(G)$.

- $\operatorname{rk} M_\varphi \geq \operatorname{rk} Z(G,k)$.
- $\operatorname{ed}_k G \leq \dim \varphi - (\operatorname{rk} M_\varphi - \operatorname{rk} Z(G,k))$.

Outline

Central extensions

- Let $G = H \times A$ a constant finite alg. group with A abelian
- Assume that $\zeta_{\exp A} \in k$ and H has a faithful completely reducible H-module W.
- Then G has a faithful G-module of the form

$$V = W \oplus k_{\chi_1} \oplus \cdots \oplus k_{\chi_r},$$

  for characters $\chi_i \in \mathrm{Hom}(G,k)$, $r = \mathrm{rk}\, A$.
- Let $\varphi\colon \mathbb{A}(W) \dashrightarrow \mathbb{A}(W)$ be a multihomogeneous minimal faithful covariant for H.
- $\Phi\colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(V), \quad (w, t_1, \dots, t_r) \mapsto (\varphi(w), t_1, \dots, t_r).$
- Φ is a faithful multihomogeneous G-covariant and $\mathrm{rk}\, M_\Phi = \mathrm{rk}\, M_\varphi + r \overset{Th}{=} \mathrm{rk}\, Z(H,k) + r.$

$$\begin{aligned}\mathrm{ed}_k\, G &\overset{Th}{\leq} \dim \Phi - (\mathrm{rk}\, M_\Phi - \mathrm{rk}\, Z(G,k)) \\ &= \mathrm{ed}_k\, H - \mathrm{rk}\, Z(H,k) + \mathrm{rk}\, Z(G,k).\end{aligned}$$

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- The reverse inequality also holds!
- It works more generally for central extensions $1 \to A \to G \to H \to 1$ with $A \cap [G,G] = \{e\}$.
- Special cases by Buhler/Reichstein (1997), Kang (2006), Kraft/Schwarz (2007).

Generalization of M. Florence's twisting technique

Setting

- $V = \bigoplus_{i=1}^{n} V_i$ a completely reducible faithful G-module.
- Let $\mathrm{PP}(V) := \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_n)$.
- $\pi \colon \mathbb{A}(V) \dashrightarrow \mathrm{PP}(V)$ the rational quotient map.

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Let $\varphi\colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(V)$ be a faithful multihomogeneous covariant.

- Get a rational map $\psi\colon\ \mathrm{PP}(V) \dashrightarrow \mathrm{PP}(V)$ fitting into a commutative square:

$$\begin{array}{ccc} \mathbb{A}(V) & \overset{\varphi}{\dashrightarrow} & \mathbb{A}(V) \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi} \\ \mathrm{PP}(V) & \overset{\psi}{\dashrightarrow} & \mathrm{PP}(V) \end{array}$$

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- Let $C \subseteq Z(G)$ act trivially on $\mathrm{PP}(V)$; ψ is G/C-equivariant
- Twist $\psi\colon \mathrm{PP}(V) \dashrightarrow \mathrm{PP}(V)$ with a G/C-torsor $E \to \operatorname{spec} K$

$$\rightsquigarrow {}^E\psi\colon \mathrm{SB}(A_1) \times \cdots \times \mathrm{SB}(A_n) \dashrightarrow \mathrm{SB}(A_1) \times \cdots \times \mathrm{SB}(A_n),$$

where $A_i = {}^E(\operatorname{End} V_i \otimes_k K)$ is some central simple K-algebra.

Generalization of M. Florence's twisting technique

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  where $A_i = {}^E(\operatorname{End} V_i \otimes_k K)$ is some central simple K-algebra.

- $$\dim \varphi - \operatorname{rk} C \geq \dim \psi \geq \dim {}^E\psi$$

  $$\geq \min \left\{ \dim \alpha \mid \alpha\colon \prod_j \mathrm{SB}(A_j) \dashrightarrow \prod_j \mathrm{SB}(A_j) \right\}$$

  $$\overset{K.-M.}{=} \text{canonical dim. of a finite subgroup of } \mathrm{Br}(K)$$

Conclusion

- G an algebraic group admitting a completely reducible faithful representation
- For any $C \subseteq Z(G)$ acting trivially on $\mathrm{PP}(V)$ and G/C-torsor $E \to \operatorname{spec} K$ we have a lower bound

$$\mathrm{ed}_k\, G + \dim G - \mathrm{rk}\, C \overset{(*)}{\geq} \text{canonical dim. of a sg. } D(C,E) \subseteq \mathrm{Br}(K)$$

- $D(C,E)$ is finite and can be described explicitly with maps from Galois cohomology
- If $D(C,E)$ is a p-group (or generated by an element of index 6) its canonical dimension has been computed by Karpenko and Merkurjev (and Colliot-Thélène)
- A stack-theoretic proof of $(*)$ was given by Karpenko and Merkurjev in case C is a p-group.