What is a Faithful Group?

Characterization and Application

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Outline

Historical roots of the Problem

In the Notes to "Theory of Groups of Finite Order" (1911) W. Burnside asked:

What groups of finite order are "simply isomorphic with irreducible groups of linear transformations".

G finite group, V complex representation of G

Definition

G is called faithful, if it has a faithful irreducible representation over $\mathbb{C}$.

K. Shoda (1930), L. Weisner (1939), M. Tazawa (1940), R. Kochendörfer (1948), Gaschütz (1954) treated the problem to characterize faithful groups.

Examples

Basic examples:

- Cyclic groups are faithful.
- Symmetric and alternating groups are faithful.
- Simple groups are faithful.
- $S_n \times S_n$ (for $n > 2$) is faithful.
- Non-cylic abelian groups are not faithful.

Schur

Lemma (Schur)

V irreducible $\Rightarrow \mathrm{End}_G(V) = \mathbb{C} \cdot \mathrm{id}_V$.

Lemma

If G is faithful then $Z(G)$ is cylic.

Proof.

Let $\rho : G \to \mathrm{Gl}(V)$ be a faithful irr. repr. of G. Then
$Z(G) \simeq \rho(Z(G)) \subseteq \mathrm{End}_G(V) = \mathbb{C} \cdot \mathrm{id}_V$, so
$Z(G)$ is isomorphic to a finite subgroup of $\mathbb{C}^*$, hence cyclic. □

So if $Z(G)$ is not cyclic then G cannot be faithful.

Proposition

$G_1, \dots, G_k$ faithful. Then $G = G_1 \times \cdots \times G_k$ is faithful $\Leftrightarrow Z(G)$ is cyclic $\Leftrightarrow n_1 := |Z(G_1)|, \dots, n_k := |Z(G_k)|$ are coprime.

proof of (3) $\Rightarrow$ (1).

If V_i are faithful irr. repr. of G_i, then the repr. $V = V_1 \otimes \cdots \otimes V_k$ $\big($where $(g_1, \dots, g_k)(v_1 \otimes \cdots \otimes v_k) = (g_1 v_1) \otimes \cdots \otimes (g_k v_k)\big)$ of G is

- irreducible: $\langle \chi_G, \chi_G \rangle = \prod \langle \chi_{G_i}, \chi_{G_i} \rangle = 1$
- faithful: Let $g = (g_1, \dots, g_k) \in \ker \rho_V$. Then for $i = 1 \dots k$:

$$\forall v_i \in V_i : g_i v_i \in \mathbb{C}^* v_i$$
$$\Rightarrow \rho_{V_i}(g_i) \in Z(\rho_{V_i}(G_i)) \simeq Z(G_i).$$

  So $\rho_{V_i}(g_i) = \lambda_i \mathrm{id}_{V_i}$ where $\lambda_i \in \mu_{n_i} \subset \mathbb{C}^*$ and $\prod_i \lambda_i = 1$. But $\mu_{n_1} \times \cdots \times \mu_{n_k} \xrightarrow{can} \mu_{\prod n_i}$ is injective. Thus $\lambda_i = 1, \forall i$ and $g = e$.

□

Non-faithful groups

Is every group with cyclic center faithful? No!

Non-faithful, trivial center (Burnside)

$G = G_{\text{Burnside}} = (S_3 \times S_3) \cap A_6 = \langle (123), (456), (12)(45) \rangle$, $|G| = 18$.
$Z(G) = \{(1)\}$, G has four 2-dim. and two 1-dim. irr. representations, none of which is faithful.
$N_1 = \langle (123) \rangle, N_2 = \langle (456) \rangle, N_3 = \langle (123)(456) \rangle, N_4 = \langle (123)(465) \rangle$. 2-dim. irr. repr. of G corresponds to 2-dim irr. repr. of $G/N_i \simeq S_3$.

There are 7 groups of order < 100 non-faithful groups with trivial center (GAP), of which G_{Burnside} is the smallest example.

Outline

Minimal normal subgroups

Notations:

$\text{minN}(G) := \{N \subseteq G | N$ minimal nontrivial normal in $G\}$
$\text{irr}(G)$:= set of [isomorphism classes of] irreducible repr. of G.
$\text{faith}(G)$:= set of [isomorphism classes of] faithful irr. repr. of G.
$\mathfrak{C}_N$:= set of [isomorphism classes of] irr. repr. V for which $N \subseteq \ker \rho_V$.
$t \equiv t(G) := |\text{minN}(G)|$, $N_1, \ldots, N_t$ the elements of $\text{minN}(G)$.

- V non-faithful repr. $\Leftrightarrow \exists N \in \text{minN}(G), N \subseteq \ker \rho_V$.
- So we have $\text{faith}(G) = \text{irr}(G) \setminus \bigcup_{N \in \text{minN}(G)} \mathfrak{C}_N$.

$$|\text{faith}(G)| = |\text{irr}(G)| + \sum_{r=1}^{t} (-1)^r \sum_{i_1 < \cdots < i_r} |\mathfrak{C}_{N_{i_1}} \cap \cdots \cap \mathfrak{C}_{N_{i_r}}|$$

further..

From the last slide:

$$|\mathrm{faith}(G)| = |\mathrm{irr}(G)| + \sum_{r=1}^{t} (-1)^r \sum_{i_1 < \cdots < i_r} |\mathfrak{C}_{N_{i_1}} \cap \cdots \cap \mathfrak{C}_{N_{i_r}}|$$

Note that $\mathfrak{C}_{N_{i_1}} \cap \cdots \cap \mathfrak{C}_{N_{i_r}} = \mathfrak{C}_{N_{i_1} \cdots N_{i_r}}$.
Moreover $|\mathfrak{C}_N| = |\mathrm{irr}(G/N)|$ for any $N \triangleleft G$.
By classical results: $|\mathrm{irr}(H)| =: h_H =$ #conjugacy classes of H.

First group intrinsic interpretation:

G is faithful if and only if $\quad h_G + \sum_{r=1}^{t} (-1)^r \sum_{i_1 < \cdots < i_r} h_{G/N_{i_1} \cdots N_{i_r}} \neq 0.$

Outline

Definition and remarks

Definition

- $\mathrm{soc}(G) := \langle N | N \in \mathrm{minN}(G) \rangle$
- $\mathrm{soc}^{\mathrm{Ab}}(G) := \langle N | N \in \mathrm{minN}(G), N \text{ abelian} \rangle$
- $\mathrm{soc}^{\mathrm{nAb}}(G) := \langle N | N \in \mathrm{minN}(G), N \text{ non-abelian} \rangle$

Lemma

- *Let $S \subseteq \mathrm{minN}(G)$. Then there exist $N_1, \ldots, N_k \in S$ such that $\langle \cup S \rangle = N_1 \times \cdots \times N_k$.*
- *If $S \subseteq T \subseteq \mathrm{minN}(G)$, then there exists $S' \subseteq T \setminus S$ such that $\langle \cup T \rangle = \langle \cup S \rangle \times \langle \cup S' \rangle$.*

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Lemma

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Proof of 1st part.

By induction on $n := |S|$. Cases $n = 0, 1$ OK, so let $N \in S, S' := S \setminus \{N\}$.
By induction: $\exists N_1, \ldots, N_k \in S' : \langle \cup S' \rangle = N_1 \times \cdots \times N_k$. The subgroup $\langle \cup S' \rangle \cap N \subseteq G$ is normal. By minimality of N either
$N \subseteq \langle \cup S' \rangle$ and thus $\langle \cup S \rangle = \langle \cup S' \rangle$ or
$\langle \cup S' \rangle \cap N = \{e\}$ and then $\langle \cup S \rangle = \langle \cup S' \rangle \times N$. □

In particular the Lemma applies to $\mathrm{soc}(G), \mathrm{soc}^{\mathrm{Ab}}(G), \mathrm{soc}^{\mathrm{nAb}}(G)$.

More about $\mathrm{soc}(G)$

Lemma

- $soc^{nAb}(G) = N_1 \times \cdots \times N_k$ *where* $N_1, \ldots, N_k$ *are precisely the non-abelian elements of* $\mathrm{minN}(G)$.
- $\mathrm{soc}(G) = soc^{Ab}(G) \times soc^{nAb}(G)$.
- *Every* $N \in \mathrm{minN}(G)$ *is a product of (isomorphic) simple groups.*
- $\mathrm{soc}(G) = G$ *iff* G *is a product of simple groups.*
- $\mathrm{soc}(G_1 \times G_2) = \mathrm{soc}(G_1) \times \mathrm{soc}(G_2)$.
- *Every* $N \triangleleft G$, $N \subseteq \mathrm{soc}(G)$ *has a complement* $N' \triangleleft G$, $N \subseteq \mathrm{soc}\, G$ *(i.e.* $N \cdot N' = \mathrm{soc}(G)$ *and* $N \cap N' = \{e\}$*).*
- *Every* $N \triangleleft G$, $N \subseteq \mathrm{soc}(G)$ *is of the form* $\langle \cup S \rangle$ *where* $S \subseteq \mathrm{minN}(G)$.

Proof.

What do you want to see? □

Examples for the socle

Examples

- $G = S_4$: $\mathrm{minN}(G) = \{V\}$, where V is the Klein four-group, $\mathrm{soc}(G) = \mathrm{soc}^{\mathrm{Ab}}(G) = V$.
- $G = S_n$ for $n \geq 5$: $\mathrm{minN}(G) = \{A_n\}$, $\mathrm{soc}(G) = \mathrm{soc}^{\mathrm{nAb}}(G) = A_n$.
- $G = D_{2\cdot 4}$: $\mathrm{minN}(G) = \{Z(G)\}$, $\mathrm{soc}(G) = \mathrm{soc}^{\mathrm{Ab}}(G) = Z(G) \simeq C_2$.
- $G = G_{\mathrm{Burnside}} = (S_3 \times S_3) \cap A_6$:
  $\mathrm{minN}(G) = \{\langle(123)\rangle, \langle(456)\rangle, \langle(123)(456)\rangle, \langle(123)(465)\rangle\}$,
  $\mathrm{soc}(G) = \mathrm{soc}^{\mathrm{Ab}}(G) = C_3 \times C_3$.
- $G = C_6 \ltimes ((C_2)^2 \times C_9 \times (A_5)^2)$, where a generator of C_6 permutes the nontrivial elements of $(C_2)^2$ cyclically, sends a generator of C_9 to its square and interchanges the components of $(A_5)^2$:
  $\mathrm{minN}(G) = \{(C_2)^2, C_3, \mathrm{diag}(A_5)^2\}$,
  $\mathrm{soc}^{\mathrm{Ab}}(G) = (C_2)^2 \times C_3, \mathrm{soc}^{\mathrm{nAb}}(G) = \mathrm{diag}(A_5)^2 \simeq A_5$.

Outline

Lattices

A lattice $(V, \cap, \cup)$ is a set V together with two *commutative, associative* inner binary operations $\cup, \cap$ with absorption law:

$$u \cap (u \cup v) = u, \quad u \cup (u \cap v) = u \text{ for all } u, v \in V.$$

If furthermore the distributive law holds:

$$u \cap (v \cup w) = (u \cap v) \cup (u \cap w),$$

then V is called a *distributive lattice*.

Examples

- The subsets of a set with intersection and union.
- The subgroups of G with $\cap$ and $\langle \bullet \cup \bullet \rangle$ of which the normal subgroups and the normal subgroups contained in $\mathrm{soc}(G)$ form a sublattice.
- The submodules of a module (or ideals of a ring) with $\cap$ and $+$.

Inclusion/exclusion principle

Let $(V, \cap, \cup)$ be a distributive lattice and $f\colon V \to \mathbb{N}_0$ such that $f(u \cup v) = f(u) + f(v) - f(u \cap v), \forall u, v \in V$. Then:

Lemma

- $f(u_1 \cup \cdots \cup u_n) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{i_1 < \cdots < i_r} f(u_{i_1} \cap \cdots \cap u_{i_r})$.
- *If V is bounded and each element in V has a complement u^c: $(u \cup u^c = \cup V, u \cap u^c = \cap V)$, then* $f((u_1 \cup \cdots \cup u_n)^c) = f(\cup V) + \sum_{r=1}^{n} (-1)^{r} \sum_{i_1 < \cdots < i_r} f(u_{i_1} \cap \cdots \cap u_{i_r})$

Proof.

Easy induction on n. □

We will apply it to the lattice of two-sided ideals in the group algebra $\mathbb{C}[G]$.

The group algebra

Definition

Let $\mathbb{C}[G]$ denote the group algebra and for $N \triangleleft G$:

$$A[G/N] := \left\{ \sum_g c_g g \in \mathbb{C}[G] : \ c_g = c_{gn} \forall n \in N \right\} \simeq \mathbb{C}[G/N].$$

$A[G/N] \subseteq \mathbb{C}[G]$ is a two-sided ideal. We have $\dim_{\mathbb{C}} A[G/N] = (G : N)$. For a group H the group-algebra $\mathbb{C}[H]$ is a semi-simple $\mathbb{C}$-algebra,

$$\mathbb{C}[H] \simeq M_{k_1}(\mathbb{C}) \times \cdots \times M_{k_r}(\mathbb{C})$$

where $r = |\mathrm{irr}(H)|$, $k_i = \dim V_i$ if $V_1, \ldots, V_r$ are [the classes of] irr. repr. of H. The center $Z(\mathbb{C}[H]) \simeq \mathbb{C}^r$ has dimension $r = |\mathrm{irr}(H)| = h_H$ and in particular $\dim_{\mathbb{C}} Z(A[G/N]) = h_{G/N}$.

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Observe that $A[G/N_{i_1} \cdots N_{i_r}] = A[G/N_{i_1}] \cap \cdots \cap A[G/N_{i_r}]$.
The map

$$\text{TwoSidedIdeals}(\mathbb{C}[G]) \longrightarrow \text{Ideals}(Z(\mathbb{C}[G])), \quad I \mapsto Z(I)$$

is an isomorphism of lattices. Thus:
$h_{G/N_{i_1} \cdots N_{i_r}} = \dim Z(A[G/N_{i_1} \cdots N_{i_r}]) = \dim Z(A[G/N_{i_1}]) \cap \cdots \cap Z(A[G/N_{i_r}])$.

Outline

Formulation of the Theorem

Theorem (Gaschütz)

The following statements are equivalent:

1. *G is faithful.*
2. $\exists g \in \mathrm{soc}(G) : \langle C^G(g) \rangle = \mathrm{soc}(G)$.
3. $\exists h \in soc^{Ab}(G) : \langle C^G(h) \rangle = soc^{Ab}(G)$.

$(2) \Leftrightarrow (3)$.

Let $g = (h, h') \in \mathrm{soc}(G) = \mathrm{soc}^{\mathrm{Ab}}(G) \times \mathrm{soc}^{\mathrm{nAb}}(G)$ such that $\langle C^G(g) \rangle = \mathrm{soc}(G)$. Then $\langle C^G(h) \rangle = \mathrm{soc}^{\mathrm{Ab}}(G)$.

Conversely let h be as above. Let $h' = (n_1, \dots, n_k) \in \mathrm{soc}^{\mathrm{nAb}}(G) = N_1 \times \dots \times N_k$ be such that $n_i \neq e_{N_k}$. Fix i and $n' \in N_i$ with $[n', n_i] \neq e_{N_i}$ and set $g = (h, h')$. We have $e \neq [n', n_i] = [n', g] \in \langle C^G(g) \cap N_i$. Since N_i is minimal, we have $N_i \subseteq C^G(g)$. So $\mathrm{soc}^{\mathrm{Ab}}(G) \times \{e\} \subseteq \langle C^G(g) \rangle$ and thus $\mathrm{soc}^{\mathrm{Ab}}(G) = \langle C^G(h) \rangle$. □

in terms of the group algebra

Our first characterization for a group to be faithful:

$$h_G + \sum_{r=1}^{t} (-1)^r \sum_{i_1 < \cdots < i_r} h_{G/N_{i_1} \cdots N_{i_r}} \neq 0$$

translates to the condition:

$$\dim Z(\mathbb{C}[G]) + \sum_{r=1}^{t} (-1)^r \sum_{i_1 < \cdots < i_r} \dim Z(A[G/N_{i_1}]) \cap \cdots \cap Z(A[G/N_{i_r}]) \neq 0.$$

By the exclusion principle for the distributive(!) lattice Ideals$(Z(\mathbb{C}[G]))$ the sum above is: $\dim Z(\mathbb{C}[G]) - \dim Z(A[G/N_1]) + \cdots + Z(A[G/N_t])$.

So G is faithful if and only if

- the $Z(A[G/N])$ for $N \in \mathrm{minN}(G)$ do not generate $Z(\mathbb{C}[G])$ or iff
- the $A[G/N]$ for $N \in \mathrm{minN}(G)$ do not generate the group algebra $\mathbb{C}[G]$.

in terms of quotients of the socle

Translating back and using $\dim A[G/N] = \dim \mathbb{C}[G/N] = (G : N)$, that is the case if and only if

$$
\begin{aligned}
0 &\neq \dim \mathbb{C}[G] + \sum_{r=1}^{t} (-1)^r \sum_{i_1 < \cdots < i_r} \dim A[G/N_{i_1}] \cap \cdots \cap A[G/N_{i_r}] \\
&= \dim \mathbb{C}[G] + \sum_{r=1}^{t} (-1)^r \sum_{i_1 < \cdots < i_r} \dim A[G/N_{i_1} \cdots N_{i_r}] \\
&= |G| + \sum_{r=1}^{t} (-1)^r \sum_{i_1 < \cdots < i_r} (G : N_{i_1} \cdots N_{i_r}) \\
&= (G : \mathrm{soc}(G)) \cdot \left(|\mathrm{soc}(G)| + \sum_{r=1}^{t} (-1)^r \sum_{i_1 < \cdots < i_r} (\mathrm{soc}\, G : N_{i_1} \cdots N_{i_r}) \right)
\end{aligned}
$$

Let V be the lattice of G-normal subgroups in $\mathrm{soc}(G)$. Since V is completely reducible there exists a anti-automorphism $\alpha\colon V \to V$. Let $L_i := \alpha(N_i)$ which is maximal non-trivial in the lattice.
We have $|L_{i_1} \cap \cdots \cap L_{i_r}| = |\alpha(N_{i_1} \cdots N_{i_r})| = (\mathrm{soc}(G) : N_{i_1} \ldots N_{i_r})$, so

$$\begin{aligned} 0 \neq |\mathrm{soc}(G)| &+ \sum_{r=1}^{t} (-1)^r \sum_{i_1 < \cdots < i_r} (\mathrm{soc}\, G : N_{i_1} \cdots N_{i_r}) \\ = |\mathrm{soc}(G)| &+ \sum_{r=1}^{t} (-1)^r \sum_{i_1 < \cdots < i_r} |L_{i_1} \cap \cdots \cap L_{i_r}| \\ = |\mathrm{soc}(G) &\setminus (L_1 \cup \cdots \cup L_t)| \end{aligned}$$

The elements of $\mathrm{soc}(G) \setminus (L_1 \cup \cdots \cup L_t)$ are precisely those $g \in \mathrm{soc}(G)$ which generate all of $\mathrm{soc}(G)$. That finishes the proof.

Outline

Definitions

- V, W faithful (complex) representations of G (i.e. ρ_V and ρ_W are injective).
- A rational covariant $\varphi\colon V \to W$ is a rational map between V and W (considered as affine varieties), *equivariant* w.r.t. the G-action ($\varphi(gv) = g\varphi(v)$).

Definition

- φ is called faithful if G acts faithfully on $\operatorname{im}\varphi$.
- $\dim\varphi :=$ dimension (as affine variety) of the Zariski-closure of $\operatorname{im}\varphi$.
- $\operatorname{edim} G := \min\{\dim\varphi \mid \varphi\colon V \to W \text{ rational covariant }\}$
- $\operatorname{covdim} G := \min\{\dim\varphi \mid \varphi\colon V \to W \text{ regular covariant }\}$

Since every regular map is also rational: $\operatorname{edim} G \leq \operatorname{covdim} G$.
For any faithful repr. W: $\operatorname{covdim} G \leq \dim W$. Equality holds e.g. for abelian groups (Buhler/Reichstein), p-groups (Karpenko/Merkurjev)

Remark: The definition of edim G and covdim G are *independent* of V and W:

Lemma

Let V, W be faithful repr. of G.

- $\exists U \subseteq V$ *dense, open* : $G_v = \{e\}$.
- *If $v \in V, w \in W$ have trivial stabilizer then there exists a (faithful) covariant $\psi\colon V \to W$ with $\psi(v) = w$.*

Proof.

- Consider $V_g := \{v \in V\colon gv = v\} \subset V$ for $g \neq \{e\}$, which is a strict closed subset. So $\bigcup_g V_g \neq V$ since V is topologically irreducible. So $U := V \setminus \bigcup V_g = \{v \in V : G_v = \{e\}\}$ is dense open in V.
- Let $f \in \mathscr{O}(V)$ with $f(v) = 1$, $f(gv) = 0$ for $g \neq e$. Then $\psi\colon V \to W : v \mapsto \sum_{g \in G} f(g^{-1}v) \cdot gw$ is equivariant and $\psi(v) = w$.

□

Covariant vs. essential dimension

Proposition

$(\operatorname{edim} G \leq) \operatorname{covdim} G \leq \operatorname{edim} G + 1$.

Proof.

- Let $\varphi\colon V \to W$ be a faithful rational covariant of $\dim \varphi = \operatorname{edim} G$.
- There exists $0 \neq f \in \mathcal{O}(V)$ such that $f\varphi$ is regular.
- Replace f by $f' \in \mathcal{O}(V)^G$, $f'(x) = \prod_{g \in G} f(g^{-1}x)$.
- $\psi = f'\varphi$ is a regular covariant, $\dim \psi \leq \dim \varphi + 1$.
- ψ is faithful because for $G_{\varphi(v)} = \{e\}$, $f'(v) \neq 0$ (which holds on a dense open subset of V) $G_{\psi(v)} = \{e\}$.

□

Outline

Lemma

If G is faithful, then there exists a faithful minimal (regular/rational) covariant which is homogeneous.

Idea of the proof.

Let V, W be faithful repr. of G, where W is irreducible. Replace a minimal faithful regular covariant $\varphi\colon V \to W$ by its highest degree part $\varphi_{\max}\colon V \to W$.

- $\dim \varphi_{\max} \leq \dim \varphi$ (not trivial)
- $\{0\} \neq \operatorname{im} \varphi_{\max} \subseteq W$ is G-stable and spans W linearly, since W is irreducible.
- If $g \in G$ acts trivially on $\operatorname{im} \varphi_{\max}$ then it is trivial on W, so $\varphi_{\max}$ is faithful.

□

Theorem (Kraft, Schwarz)

If G is a non-trivial faithful, then $\operatorname{covdim} G = \operatorname{edim} G + 1 \Leftrightarrow Z(G) = \{e\}$. *Otherwise* $\operatorname{edim} G = \operatorname{covdim} G$.

Proof.

- If φ is a homogeneous regular covariant, then $\psi\colon V \xrightarrow{\varphi} V \to \mathbb{P}(V)$ is a rational covariant of dimension $\dim\psi \leq \dim\varphi - 1$. If $Z(G) = \{e\}$ and if φ is faithful, then ψ is faithful, too. So if $\dim\varphi = \operatorname{covdim} G$ then $\operatorname{edim} G \leq \dim\varphi - 1 \leq \operatorname{covdim} G - 1$.
- If φ is a homogeneous rational covariant and $0 \neq f' \in \mathcal{O}(V)^G$ s.t. $\psi = f'\varphi$ is regular, then $\operatorname{im}\psi \subseteq \operatorname{im}\varphi$ and thus $\dim\psi \leq \dim\varphi$.

□

Outline

Results from Kraft, Schwarz

- $\operatorname{covdim} G = 1 \Rightarrow G$ is cyclic.
- $\operatorname{covdim} G \leq 2 \Rightarrow G$ has a faithful 2-dim. representation.

Important step for the proof:

If G is non-faithful of $\operatorname{covdim} G = 2$, then G is abelian!
Assume the opposite and take a minimal counter-example G. Every strict subgroup $H \subset G$ is either commutative or faithful. If $\operatorname{soc}^{\mathrm{Ab}}(G) \subseteq H$, then H must be commutative by the following Lemma. Such groups are quite special.

Lemma

Let $H \subseteq G$ be finite groups. If H is faithful and $\operatorname{soc}^{\mathrm{Ab}}(G) \subseteq H$, then G is faithful, too.

The proof makes use of Gaschuetz' characteization of faithful groups.

Proof with help of Gaschuetz' criterion.

- Since $\mathrm{soc}^{\mathrm{Ab}}(G) \cap \mathrm{soc}^{\mathrm{Ab}}(H) \subseteq \mathrm{soc}^{\mathrm{Ab}}(H)$ are H-normal subgroups contained in $\mathrm{soc}(H)$:
  $\exists N \lhd H$ s.t. $N \times \mathrm{soc}^{\mathrm{Ab}}(G) \cap \mathrm{soc}^{\mathrm{Ab}}(H) = \mathrm{soc}^{\mathrm{Ab}}(H)$.
- For $h = (n, g) \in \mathrm{soc}^{\mathrm{Ab}}(H)$ with $\langle C_h^H \rangle = \mathrm{soc}^{\mathrm{Ab}}(H)$:
  $\langle C_g^H \rangle = \mathrm{soc}^{\mathrm{Ab}}(G) \cap \mathrm{soc}^{\mathrm{Ab}}(H)$.
- C_g^G then generates $\mathrm{soc}^{\mathrm{Ab}}(G)$, since $\langle C_g^G \rangle$ intersects every $N' \in \mathrm{minN}(G)$ nontrivially.

□

W.Burnside *Theory of Groups of Finite Order* (1911) Ed. Cambridge, Note F 476-478

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H.Kraft, G.W.Schwarz *Compression of finite group actions and covariant dimension* (2007), J. Algebra 313:268-291.