Small Faithful Representations and the Essential Dimension of Finite Groups

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- G a finite group
- k an arbitrary field
- V a faithful representation $\rho\colon G \hookrightarrow \mathrm{Gl}(V)$.

Basic Goal:

Understand faithful V with $\dim V$ as small as possible!

Definition

$\mathrm{rdim}_k\, G := \min\{\dim V | V \text{ faithful rep. of G over k}\}$.

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Examples

- Abelian groups: $A = \mathbb{Z}/m_1\mathbb{Z} \times \ldots \times \mathbb{Z}/m_r\mathbb{Z}$ where $m_1|m_2|\ldots|m_r = \exp(A)$
  If $\zeta_{\exp(G)} \in k$ $\operatorname{rdim}_k(A) = r =: \operatorname{rk} A$.
- $\mathbb{Z}/p^m\mathbb{Z}$: (Florence) Let $1 \leq s \leq m$ with $\zeta_{p^s} \in k$ but $\zeta_{p^{s+1}} \notin k$.
  Then $\operatorname{rdim}_k \mathbb{Z}/p^m\mathbb{Z} = p^{m-s}$, $V = \operatorname{ind}_{\mathbb{Z}/p^s\mathbb{Z}}^{\mathbb{Z}/p^m\mathbb{Z}} k_{\zeta_{p^s}}$.

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- Symmetric group S_n, $n \geq 3$:
  $V_n = \{(v_1, \ldots, v_n) \in k^n \mid \sum v_i = 0)\}$.
  If char $k = p$ and $p|n$: V_n is not irreducible, but $V_n/(k \cdot (1, 1, \ldots, 1))$ is.
  $\operatorname{rdim}_k(G) = n - 1$ or $n - 2$, respectively.

Completely reducible representations (c.r. rep.)

Consider only rep. $V = \bigoplus_{i=1}^{n} V_i$ where V_i irreducible.

Warning

Sometimes all c.r. rep. are non-faithful! This happens only if G has a nontrivial normal p-subgroup, char $k = p > 0$.

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Reduction to c.r. case

arbitrary rep. $V \rightsquigarrow$ c.r. rep. V_{ass}:
$V_{ass} = \bigoplus_{i=1}^{n} V_{i+1}/V_i$ where $V =: V_n \supsetneq V_{n-1} \supsetneq \ldots \supsetneq V_0 = 0$ is a composition series.
V faithful $\Rightarrow V_{ass}$ faithful.

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V faithful $\Rightarrow V_{\text{ass}}$ faithful.

Question:

How many irreducible components are needed for a faithful rep.?

Behaviour of $\text{rdim}_k(G)$

- k'/k field extension $\Rightarrow \text{rdim}_{k'}(G) \leq \text{rdim}_k(G)$
- $G \subseteq G'$ subgroup $\Rightarrow \text{rdim}_k(G) \leq \text{rdim}_k(G')$
- $\text{rdim}_k(G_1 \times G_2) \leq \text{rdim}_k(G_1) + \text{rdim}_k(G_2)$ (take $V_1 \oplus V_2$).
- $N \lhd G \overset{?}{\Rightarrow} \text{rdim}_k(G/N) \leq \text{rdim}_k(G)$ No!!

Example: (Brosnan, Reichstein, Vistoli)

- $G = Q \times Q \times Q$ where $Q = \{\pm 1, \pm i, \pm j, \pm k\}$,
- $N = \{(1,1,1), (1,-1,-1), (-1,1,-1), (-1,-1,1)\} \subset Z(G)$.
- Assume $\zeta_4 \in k$: $\text{rdim}_k(G) \leq 3\,\text{rdim}_k\, Q = 6$.
- G/N has 2^6 one-dim. rep. and one irr. rep. of dim. 2^3.
- $\text{rdim}_k(G/N) = 2^3 = 8$.

An important subgroup: the socle of G

- $\mathrm{minN}(G) := \{N \lhd G \mid N \text{ minimal nontrivial normal subgroup}\}$.
- $\mathrm{soc}(G) := \langle N \mid N \in \mathrm{minN}(G)\rangle$
- $\mathrm{soc}^{\mathrm{Ab}}(G) := \langle N \mid N \in \mathrm{minN}(G), A \text{ abelian.}\rangle$
- Both $\mathrm{soc}(G), \mathrm{soc}^{\mathrm{Ab}}(G) \lhd G$ are normal; $\mathrm{soc}^{\mathrm{Ab}}(G)$ is abelian.

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Structure

$$\mathrm{soc}(G) = \mathrm{soc}^{\mathrm{Ab}}(G) \times N_1 \times \ldots \times N_t$$

where $N_1, \ldots, N_t$ are all the non-abelian el. of $\mathrm{minN}(G)$.

$$\mathrm{soc}^{\mathrm{Ab}}(G) = A_1 \times \ldots \times A_r$$

for some (abelian) el. of $\mathrm{minN}(G)$, $A_i \simeq (\mathbb{Z}/p_i\mathbb{Z})^{m_i}$.

More about the socle

Examples:

- $G = S_4$: $\operatorname{soc} G = V_4 \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ Klein four-group.
- $G = S_n, n \geq 5$: $\operatorname{soc} G = A_n$
- $G = (S_3 \times S_3) \cap A_6$: $\operatorname{soc} G = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ (Burnside)

The abelian socle $\operatorname{soc}^{\mathrm{Ab}}(G)$ is a $\mathbb{Z}G$-module via the conjugation action of G.

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- $G = S_n, n \geq 5$: $\operatorname{soc} G = A_n$
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  $(12)(45)$ maps $x \in \operatorname{soc}(G)$ to x^{-1}, not cyclic G-module.

The abelian socle $\operatorname{soc}^{\mathrm{Ab}}(G)$ is a $\mathbb{Z}G$-module via the conjugation action of G.

Number of irreducible components

$V = \bigoplus_{i=1}^n V_i$ faithful, V_i irreducible. What is the least possible n?

Theorem

The least possible $n \in \mathbb{N}$ for faithful $V = \bigoplus_{i=1}^n V_i$ equals:

$$\max\{\mathrm{rk}_{\mathbb{Z}G}(soc^{Ab}(G)), 1\}.$$

rank = the least number of generators.

Special Case: Gaschütz (1954)

G has a faithful irr. rep. $\Leftrightarrow \mathrm{soc}^{\mathrm{Ab}}(G)$ is a cyclic $\mathbb{Z}G$-module.

Examples

- S_4 has a faithful irr. rep., $\mathrm{rk}_{\mathbb{Z}S_4}(V_4) = 1$.
- $G = (S_3 \times S_3) \cap A_6$ needs at least two irreducible components, $\mathrm{rk}_{\mathbb{Z}G}(\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}) = 2$.

Restriction to the abelian socle

- $V = \bigoplus_{i=1}^n V_i$ rep. of G with V_i irreducible
- Restrict V to $A := \mathrm{soc}^{\mathrm{Ab}}(G)$.
- $V|A = \bigoplus_{i=1}^n V_i|A$.
- $V_i|A$ is again completely reducible! (Clifford, $A \lhd G$)

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- $V_i|A$ is again completely reducible! (Clifford, $A \lhd G$)
- More precisely $V_i|A = \sum_{g \in G} gW_i = \bigoplus_{g \in S} gW_i$ where $W_i \subseteq V_i|A$ is any irreducible sub-rep. and $S \subseteq G$ is some subset;
- $gW_i \subseteq V_i|A$ carries the A-action twisted through conjugation by g^{-1}:

$$a(gw_i) = g(g^{-1}ag)w_i \quad \text{for } a \in A, g \in G.$$

If $k = \bar{k}$: Every $W_i \subseteq V_i|A$ is one-dimensional. Let $\chi_i = \chi_{W_i}$.

Lemma:

- $\ker \rho_V$ contains no abelian normal subgroup $\Leftrightarrow \langle \chi_1, \dots, \chi_n \rangle_{\mathbb{Z}G} = A^* := \mathrm{Hom}(A, k^*)$.
- In particular: $\langle \chi_1, \dots, \chi_n \rangle_{\mathbb{Z}G} = A^*$ and $\ker \rho_V$ contains no non-abelian normal subgroup $\Rightarrow V$ faithful.

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Proof: uses lattices

- $L(A) :=$ lattice of $\mathbb{Z}G$-submodules of $A := \mathrm{soc}^{\mathrm{Ab}}(G)$.
  Two operations, meet: $U_1 \cdot U_2$ and join: $U_1 \cap U_2$.
- $$\alpha\colon L(A) \to L(A^*), \quad U \mapsto \{\chi \in A^* | \chi_U = 1\}$$
  $\alpha(U_1 \cap U_2) = \alpha(U_1) \cdot \alpha(U_2), \alpha(U_1 \cdot U_2) = \alpha(U_1) \cap \alpha(U_2)$.
- anti-isomorphism, $\alpha^{-1}(C) = \{a \in A \mid \forall \chi \in C : \ \chi(a) = 1\}$.

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Proof of the Lemma

- Let $C := \langle \chi_1, \dots, \chi_n \rangle_{\mathbb{Z}G}$ which is G-invariant.
- **Claim:**
$$\alpha^{-1}(C) = \ker \rho_V \cap A = \ker \rho_{V|A}$$
or equivalently $C = \alpha(\ker \rho_{V|A})$.
- $C \subseteq \alpha(\ker \rho_{V|A})$: Each χ_i is trivial on $\ker \rho_{V|A}$, hence contained in $\alpha(\ker \rho_{V|A})$.

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- $\alpha^{-1}(C) \subseteq \ker \rho_{V|A}$: $\alpha^{-1}(C)$ is a nontrivial G-invariant subgroup of A contained in the kernel of each χ_i.
  $\Rightarrow \alpha^{-1}(C)$ is in the kernel of $\rho_V|A$, since $V_i|A = \bigoplus_{g \in S} \chi_i^{(g)}$ where $V = \bigoplus_{i=1}^n V_i$.

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- All in all $\alpha^{-1}(C) = \ker \rho_V \cap A$. In particular $C \neq A^* \Leftrightarrow \ker \rho_V \cap A \neq \{e\}$.

Reminder

Remember the Theorem: The least number n of irreducible components of a faithful representation $V = \bigoplus_{i=1}^{n} V_i$ equals $\max\{\mathrm{rk}_{\mathbb{Z}G} A, 1\}$, where $A = \mathrm{soc}^{\mathrm{Ab}}(G)$.

- Actually $\mathrm{rk}_{\mathbb{Z}G} A = \mathrm{rk}_{\mathbb{Z}G} A^*$ (see later).
- Until now: We need at least $\mathrm{rk}_{\mathbb{Z}G} A^*$ many irreducible components since $\ker \rho_V \cap A = \{e\} \overset{\text{Lemma}}{\Rightarrow} \langle \chi_1, \ldots \chi_n \rangle = A^*$.
- We don't need more components: Construct representations!

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- We don't need more components: Construct representations!

Notation: Let $N \triangleleft G$. Define $\mathrm{Rep}_N^G(W) := \{V \text{ irr. rep. of G} : V|N \supseteq W\}$ where W is an irreducible rep. of N.

Lemma (Clifford, 1937)

$\mathrm{Rep}_N^G(W) \neq \emptyset$.

In fact: V irreducible sub-rep. of $\mathrm{ind}_N^G(W) \Rightarrow V \in \mathrm{Rep}_N^G(W)$.

Construction of representation

Goal: Construct a faithful rep. with $\mathrm{rk}_{\mathbb{Z}G}\, A^*$ many irreducible components.

- Let $\langle \chi_1, \ldots, \chi_n \rangle_{\mathbb{Z}G} = A^*$.
- Then $\bigoplus_{i=1}^n k_{\chi_i}$ does not contain any minimal nontrivial abelian normal subgroup in its kernel.
- Choose for every non-abelian el. N_j of $\mathrm{minN}(G)$ some nontrivial irr. rep. W_j.
- The kernel of $W := W_1 \otimes \ldots \otimes W_t$ does not contain any non-abelian minimal normal subgroup.

Two cases:

- Case $A := \mathrm{soc}^{\mathrm{Ab}}(G) = \{e\}$: Any $\Omega \in \mathrm{Rep}^G_{\mathrm{soc}\, G}(W)$ is a faithful irr. rep. of G.
- Case $A \neq \{e\}$: Choose $\Omega_i \in \mathrm{Rep}^G_{\mathrm{soc}\, G}(k_{\chi_i} \otimes W)$; $\Omega := \bigoplus_{i=1}^n \Omega_i$ is faithful with $n = \mathrm{rk}_{\mathbb{Z}G}(A^*)$ many irr. components.

Relating the abelian socle with its character group

- Why is $\mathrm{rk}_{\mathbb{Z}G}\, A = \mathrm{rk}_{\mathbb{Z}G}\, A^*$? In general A and A^* are not G-isomorphic!
- We know: $L(A)$ and $L(A^*)$ are canonically anti-isomorphic.
- But $L(A)$ and $L(A^*)$ are also (non-canonically) isomorphic!

$$\exists \beta\colon L(A) \xrightarrow{\simeq} L(A^*) \text{ satisfying } |\beta(W)| = |W| \text{ for } W \in L(A).$$

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$$E_r(A) := \{(x_1, \ldots, x_r) \in A | \langle x_1, \ldots, x_r \rangle_{\mathbb{Z}G} = A\}.$$

- $E_r(A) = A^r \setminus \cup M^r$ where the union is taken over all maximal nontrivial elements of $L(A)$.
- The cardinality of a set $\cup Z_i$ can be expressed in terms of the cardinalities of intersections of the Z_i.
- Therefore $|E_r(A)| = |E_r(A^*)|$ (for any $r \in \mathbb{N}$).

If k is not algebraically closed

- At most that many: Let $\bar{V} = \bigoplus_{i=1}^{n} \bar{V}_i$ be a faithful rep. of G over $\bar{k}$.
  Replace each $\bar{V}_i$ by some irr. rep. V_i of G over k having $\bar{V}_i$ as a decomposition factor.
- At least that many: Let $V = \bigoplus_{i=1}^{n} V_i$ be faithful rep. over k.
- Choose $\chi_i \subseteq V_i|A \otimes \bar{k}$.
- Again: $\langle \chi_1, \ldots, \chi_n \rangle = \mathrm{Hom}(A, \bar{k}^*)$. Why?
- Key: If $V_i|A \otimes \bar{k} \supseteq \chi, \chi'$ then $\langle \chi' \rangle_{\mathbb{Z}G} = \langle \chi \rangle_{\mathbb{Z}G}$.

Representation Dimension and Essential Dimension

Recall: $\operatorname{rdim}_k G := \min\{\dim V | V$ faithful rep. of G over k$\}$.
Now: $\operatorname{edim}_k G$. Assume $k = \bar{k}$!

Definition

A covariant is a G-equivariant rational map $\Phi\colon V \to W$ where V, W are faithful rep. over k. Equivariant: $\Phi(gv) = g\Phi(v)$.

The covariant Φ is called faithful, if G acts faithfully on $\Phi(V)$.

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Definition

$\operatorname{edim}_k G = \min\{\dim \overline{\Phi(V)} \mid \Phi \text{ faithful covariant}\}$

Clearly: $\operatorname{edim}_k G \leq \operatorname{rdim}_k G$. (since id_V is a faithful covariant)

Properties of essential dimension

- $G \subseteq G'$ subgroup $\Rightarrow \operatorname{edim}_k G \leq \operatorname{edim}_k G'$
- (Also: k'/k field extension $\Rightarrow \operatorname{edim}_{k'} G \leq \operatorname{edim}_k G$)
- $\operatorname{edim}_k(G_1 \times G_2) \leq \operatorname{edim}_k G_1 + \operatorname{edim}_k G_2$
- $\operatorname{edim}_k A = \operatorname{rk} A$ if A is abelian and $\operatorname{char} k \nmid |A|$.
- For p-groups: $\operatorname{edim}_k G = \operatorname{rdim}_k G$ when $p \neq \operatorname{char} k$. (Karpenko, Merkurjev, 2007)
- Symmetric groups: $\lfloor \frac{n}{2} \rfloor \leq \operatorname{edim}_k S_n \leq n-3$ for $n \geq 5$ and $\operatorname{char} k \neq 2$.
- $\operatorname{edim}_{\mathbb{C}} S_7 = ??$

Situation of interest: $A \subseteq Z(G)$ with $A \cap [G, G] \neq \{e\}$.

Examples

- $G = H \times A$ with A abelian.
- $G = \mathbb{Z}/p^2\mathbb{Z} \ltimes \mathbb{Z}/p^2\mathbb{Z}$.
- $G = C_4 \ltimes A_4$.

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- $G = \mathbb{Z}/p^2\mathbb{Z} \ltimes \mathbb{Z}/p^2\mathbb{Z}$.
- $G = C_4 \ltimes A_4$.

Assume that G contains no nontrivial normal p-subgroup if char $k = p$.

Theorem

$\operatorname{edim}_k G = \operatorname{edim}_k G/A + \operatorname{rk} Z(G) - \operatorname{rk} Z(G)/A.$

generalizes a theorem of Buhler/Reichstein and Chang.

Ideas of the Proof

What does the Theorem have to do with representations?

Lemma:

Let $V = \bigoplus_{i=1}^{n} V_i$ be a faithful rep. Then $\exists \Phi = (\varphi_1, \ldots, \varphi_n) \colon V \to V$ minimal faithful covariant with property:

$$\Phi_j(\lambda_1 v_1, \ldots, \lambda_n v_n) = \lambda_1^{a_{1j}} \cdot \cdots \cdot \lambda_n^{a_{nj}} \Phi_j(v_1, \ldots, v_n).$$

for some matrix $A = (a_{ij}) \in \mathbb{Z}^{n \times n}$.

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Main observation:

If Φ is minimal (i.e. $\dim \overline{\varphi(V)} = \operatorname{edim} G$) then

$$\operatorname{rk} A = \operatorname{rk} Z(G).$$

In general:

$$\operatorname{edim}_k G \leq \dim \overline{\varphi(V)} - \underbrace{(\operatorname{rk} A - \operatorname{rk} Z(G))}_{\geq 0}.$$

Summary

- The least number of irreducible components for a faithful representation is given by $\max\{\mathrm{rk}_{\mathbb{Z}G}\,\mathrm{soc}^{\mathrm{Ab}}(G), 1\}$, assuming G has no nontrivial normal p-subgroup if char $k = p$.
- For central extensions with $A \cap [G, G] = 1$:

$$\mathrm{edim}\, G - \mathrm{rk}\, Z(G) = \mathrm{edim}\, G/A - \mathrm{rk}\, Z(G/A)$$

.

- Unsolved: How many decomposition factors are needed if G has a nontrivial normal p-subgroup and char $k = p$?