Multihomogenization: a new Technique in Essential Dimension

Roland Lötscher

UNI BASEL

Covariants and Essential Dimension

Essential dimension is a numerical invariant of finite groups.

Our Setting:

- G finite group, k a field.
- $\rho_V\colon G \hookrightarrow \mathrm{GL}(V)$ and $\rho_W\colon G \hookrightarrow \mathrm{GL}(W)$ faithful representations.

Basic notions:

- A *covariant* is a G-equivariant rational map $\varphi\colon V \to W$,
- equivariant: $\varphi(\rho_V(g)(v)) = \rho_W(g)(\varphi(v))$.

$$\mathrm{edim}_k G := \min\{\dim \varphi \mid \varphi \text{ faithful covariant def. over } k\},$$

- faithful: the image of φ contains points with trivial stabilizer.
- $\dim \varphi := \dim \overline{\varphi(V_{\bar{k}})} = \mathrm{trdeg}_k\, k(\varphi)$ where $k(\varphi)/k$ is the field extension generated by the coordinates of φ.

Multihomogeneous Covariants

Let $V = \bigoplus_{i=1}^m V_i$, $W = \bigoplus_{j=1}^n W_j$ faithful representations. A covariant $\varphi = (\varphi_1, \dots, \varphi_n)\colon V \to W$ is *multihomogeneous* if for some $m_{ij} \in \mathbb{Z}$:

$$\varphi_j(v_1, \dots, v_{i-1}, \lambda v_i, v_{i+1}, \dots, v_m) = \lambda^{m_{ij}} \varphi_j(v_1, \dots, v_m).$$

- $\mathbb{G}_m^m \simeq T_V \subseteq \mathrm{GL}(V), \mathbb{G}_m^n \simeq T_W \subseteq \mathrm{GL}(W)$ associated tori.
- Multihomogeneity of φ is equivalent to the identity

$$\varphi(tv) = D(t)\varphi(v) \text{ for some } D \in \mathrm{Hom}(T_V, T_W).$$

- $D_\varphi(t_1, \dots, t_m) = (t_1', \dots, t_n')$ where $t_j' = \prod_{i=1}^n t_i^{m_{ij}}$.

Multihomogenization: $\varphi \rightsquigarrow H_\lambda(\varphi)$

It is the way to produce a multihomog. covariant $H_\lambda(\varphi)\colon V \to W$ from an arbitrary covariant $\varphi\colon V \to W$.

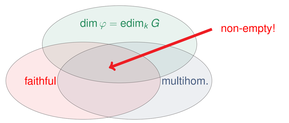
- $\lambda \in \mathrm{Hom}(\mathbb{G}_m, T_V)$ is the choice of a suitable 1-PSG (1-parameter-subgroup).
- $\mu \in \mathrm{Hom}(\mathbb{G}_m, T_W)$ is another 1-PSG, constructed from λ and φ.
- $\varphi^{(t)}\colon V \to W$ for $t \neq 0$, defined by

$$\varphi^{(t)}(v) = \mu(t)\varphi(\lambda(t)v),$$

is a family of covariants deforming $\varphi = \varphi^{(1)}$ into:

$$H_\lambda(\varphi) := \lim_{t \to 0} \varphi^{(t)}.$$

- $\dim H_\lambda(\varphi) \leq \dim \varphi$
- φ faithful $\Rightarrow$ $H_\lambda(\varphi)$ faithful.



We use Semisimple Representations.

Semisimple: A direct sum of simple sub-representations.

Our rep. $V = \bigoplus_{i=1}^m V_i$ should be faithful!

Assume $\mathrm{char}\, k =: p > 0$. G has a semisimple faithful rep. $\Leftrightarrow \nexists H \lhd G$ of order p^r, $r > 0$.

Examples in $\mathrm{char}\, k = 2$**:**

- Bad case: $G = S_4$ contains Klein group $V \lhd G$ of order 4.
- Good case: $G = S_3$. Normal subgroups are of order $1, 3, 6$.

Yoga of k-centers and Degree Matrices

$Q :=$ quaternion group, $G := Q \times \mathbb{Z}/9\mathbb{Z}$, $k := \mathbb{Q}\left[e^{\frac{\pi i}{6}}\right]$

Let $\varphi\colon V_1 \to W_1$ and $\psi\colon V_2 \oplus V_3 \to W_2$ some faithful multihomogeneous covariants for Q and $\mathbb{Z}/9\mathbb{Z}$, respectively. For example

$$\varphi(\lambda v_1) = \lambda^2 \varphi(v_1)$$
$$\psi(\lambda v_2, \mu v_3) = \lambda^5 \mu^{-2} \psi(v_2, v_3)$$

Then $M_{\varphi \times \psi} = \begin{pmatrix} 2 & 0 \\ 0 & 5 \\ 0 & -2 \end{pmatrix}$, which has rank 2.

Note: $Z(Q, k) = \{\pm 1\}$, $Z(\mathbb{Z}/9\mathbb{Z}, k) = \mathbb{Z}/3\mathbb{Z}$, hence $\mathrm{rank}\, Z(G, k) = 1 < 2 = \mathrm{rank}\, Z(Q, k) + \mathrm{rank}\, Z(\mathbb{Z}/9\mathbb{Z}, k)$.

This implies $\mathrm{edim}_k G < \dim \varphi \times \psi = \dim \varphi + \dim \psi$, even if φ and ψ are of minimal dimension among faithful covariants.

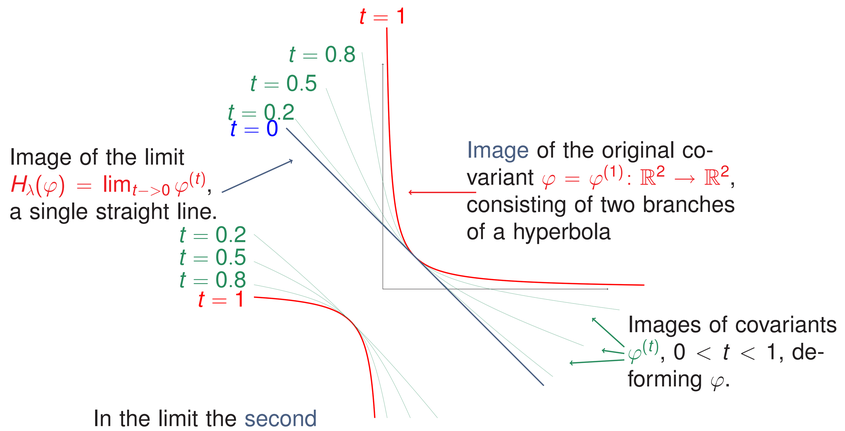
More precisely: $\mathrm{edim}_k Q = 2$, $\mathrm{edim}_k \mathbb{Z}/9\mathbb{Z} = 3$, $\mathrm{edim}_k G = 4$.

The Image of the Multihomogenization as a Limit

Start with $V = \mathbb{R}^2$ endowed with the representation of $G = \mathbb{Z}/2\mathbb{Z}$ given by $[1](x, y) = (y, x)$. The (faithful) covariant

$$\varphi\colon V \to V,\ (x, y) \mapsto \left(\frac{x}{y}, \frac{y}{x}\right)$$

is not multihomogeneous for $V = \mathbb{R}(1, 1) \oplus \mathbb{R}(1, -1)$. Let us multihomogenize it!



What does the Rank of $M_\varphi := (m_{ij}) \in M_{m \times n}(\mathbb{Z})$ tell?

Let $Z(G)$ be the center of G. The k-center is the subgroup $Z(G, k) := \{g \in Z(G) \mid k \text{ contains a primitive } (\mathrm{ord}\, g)\text{-th root of unity.}\}$

- Another description: $\rho_V(G) \cap T_V = \rho_V(Z(G, k))$.
- G-equivariance of a multihomogeneous covariant $\varphi\colon V \to W$ implies that $\rho_W(Z(G, k)) \subseteq D_\varphi(T_V)$.
- Hence $\mathrm{rank}\, M_\varphi = \dim D_\varphi(T_V) \geq \mathrm{rank}\, Z(G, k)$.

Moreover we have the following lower bound:

$$\mathrm{edim}_k G \leq \dim \varphi - (\mathrm{rank}\, M_\varphi - \mathrm{rank}\, Z(G, k)).$$

Twisting Produces Maps between Interesting Varieties.

- $V = \bigoplus_{i=1}^m V_i$ semisimple faithful rep. of G.
- $\pi\colon V \to \mathbb{P}(V) := \prod \mathbb{P}(V_i)$ canonical map into projective space.

If $\varphi\colon V \to V$ is multihomogeneous then $(\pi \circ \varphi)(tv) = \pi(D(t)\varphi(v)) = \pi \circ \varphi(v)$ and we get a commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & V \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbb{P}(V) & \xrightarrow[\exists! \psi]{} & \mathbb{P}(V) \end{array}$$

- $\psi\colon \mathbb{P}(V) \to \mathbb{P}(V)$ is a G-equivariant rational map.
- The k-center $Z := Z(G, k)$ acts trivially on $\mathbb{P}(V)$.
- $\dim \psi = \dim \varphi - \mathrm{rank}\, Z(G, k)$.

Twist:

- We can twist ψ with a G/Z-torsor E over a field extension K/k.
- The twist of $\mathbb{P}(V_i \otimes K)$ with E is the Severi-Brauer variety associated to some central simple K-algebra A_i.

Result:

- $\rightsquigarrow$ rational map ${}^E\psi\colon \prod \mathrm{SB}(A_i) \to \prod \mathrm{SB}(A_i)$.
- Upper bound: $\dim \varphi + \mathrm{rank}\, Z(G, k) \geq \dim \psi \geq \dim {}^E\psi$.

History and Outlook

Covariants have been used since the introduction of essential dimension by J. Buhler and Z. Reichstein in 1995. Multihomogenization was introduced by H. Kraft, G. Schwarz and the author in 2007. There is a more general notion of $\mathrm{edim}_k G$ for algebraic groups.

Basic questions:

- Does multihomogenization work in this setting as well?
- If yes, how do previously obtained results generalize?

Institute of Mathematics - University of Basel, Switzerland Mail: roland.loetscher@unibas.ch WWW: http://www.math.unibas.ch/~loetsche/