Multihomogeneous covariants and the essential dimension of finite groups

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November 6, 2008

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Where we are:

Motivation

- Fix some base field k (char $k \neq 2$ for the moment).
- Consider a field extension $K(x)/K$ of degree 2, $K \supseteq k$.
- Minimal polynomial $x^2 + ax + b = 0$ where $a, b \in K$.
- Complete the square: $y^2 + b' = 0$ where $y = x + \frac{a}{2}$, $b' = b - \frac{a^2}{4}$.
- $K(x) = K(y) \supset K$ is defined by one single parameter $b' \in K$.
- Similarly if char $k = 2$ and $K(x)/K$ is separable, some generator $z \in K(x)$ has min. poly. $z^2 + z + c = 0$, $c \in K$.

Conclusion: (Separable) field extensions of degree 2 over K are defined by one parameter.

Question: How many parameters are needed for (separable) field extensions of degree n?

Field extensions

- k base field
- L/K separable field extension of degree n, $K \supseteq k$.

"How many parameters?":

Number of parameters needed to define L/K := minimum of $\text{tdeg}_k\, k(a_1, a_2, \dots, a_n)$ taken over all generators y of L, where $a_1, \dots, a_n$ = coefficients of the minimal polynomial of y over K.

- $n = 2$: 1 parameter, $y^2 + a = 0$.
- $n = 3$: 1 parameter, $y^3 + ay + a = 0$.
- $n = 4$: 2 parameters, $y^4 + ay^2 + by + b = 0$.
- $n = 5$: Hermite: 2 parameters, $y^5 + ay^3 + by + b = 0$.
- $n = 6$: Joubert: 3 parameters, $y^6 + ay^4 + by^2 + cy + c = 0$, char $k \neq 2$.
- $n \geq 7$: Nobody knows!

Galois extensions

- L^{norm} a normal closure of L.
- **Fact:** Number of parameters needed to define L/K and L^{norm}/K coincide.

Reformulation:

Let G be a finite group. How many parameters are needed (in maximum) to define Galois extensions L/K over k with $\mathrm{Gal}(L/K) = G$?

Fact: The maximal number is attained for the Galois extension $k(V)/k(V)^G$ where:

- V is a faithful G-module over k
- $k(V)$ is the field of rational functions on V.

Essential dimension

V a faithful G-module (over k).

Definition (Buhler/Reichstein, 1995)

$\operatorname{edim}_k G := \min\{\operatorname{tdeg}_k E \mid k \subseteq E \subseteq k(V) \text{ faithful } G\text{-stable subfield}\}$

Note: $\operatorname{edim}_k G$ does not really depend on V!

$$\operatorname{edim}_k G \leq \min\{\dim_k V | V \text{ faithful } G\text{-module over } k\} =: \operatorname{rdim}_k G$$

G=S_n, V = canonical n-dim. permutation module, char $k \neq 2$.

- E :=subfield of $k(V) = k(x_1, \ldots, x_n)$ generated by cross-ratios $[x_i, x_j; x_k, x_l] := \frac{(x_i - x_k)(x_j - x_l)}{(x_i - x_l)(x_j - x_k)} \in k(V)$
- $\operatorname{tdeg}_k E = n - 3$ for $n \geq 3$.
- E is faithful for $n \geq 5$. In that case $\operatorname{edim}_k S_n \leq n - 3$.

Where we are:

Covariants

Definition

A covariant of G is a G-equivariant rational map

$$\varphi\colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(W)$$

where V, W are (faithful) G-modules.

φ is called faithful if the image of the generic point of $\mathbb{A}(V)$ has trivial stabilizer.

$$\begin{aligned}\dim\varphi &:= \text{ dimension of the Zariski-closure of the image of } \varphi.\\ &= \operatorname{tdeg}_k k(f_1,\dots,f_d) \text{ where } \varphi=(f_1,\dots,f_d).\end{aligned}$$

Definition (Geometric definition of essential dimension)

$\operatorname{edim}'_k G := \min\{\dim\varphi \mid \varphi \text{ faithful covariant of } G \text{ over } k\}$.

Relation of the two definitions

1. $\mathrm{edim}_k\, G := \min\{\mathrm{tdeg}_k\, E \mid k \subseteq E \subseteq k(V) \text{ faithful } G\text{-subfield}\}$
2. $\mathrm{edim}'_k\, G := \min\{\dim \varphi \mid \varphi \text{ faithful covariant of } G \text{ over } k\}$.

- $\varphi = (f_1, \dots, f_d)\colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(W)$ faithful covariant
  $\Rightarrow E := k(f_1, \dots, f_d) \subseteq k(V)$ faithful G-subfield
- $\Rightarrow \mathrm{edim}_k\, G = \min\{\mathrm{tdeg}_k\, E\} \le \min\{\dim \varphi\} = \mathrm{edim}'_k\, G$

Conversely: From a faithful G-subfield $k \subseteq E \subseteq k(V)$ one can construct a faithful covariant $\varphi\colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(W)$ with $\dim \varphi \le \mathrm{tdeg}_k\, E$.

Example ($G = S_n$ ($n \ge 5$), $E =$ field of cross ratios)

$\varphi = (f_\sigma)_{\sigma \in G}\colon \mathbb{A}(V) \to \mathbb{A}(kS_n)$ where $f_\sigma = [x_{\sigma(1)}, x_{\sigma(2)}; x_{\sigma(3)}, x_{\sigma(4)}]$.

Properties of essential dimension

- $H \subseteq G$ subgroup $\Rightarrow \operatorname{edim}_k H \leq \operatorname{edim}_k G$
- $G = G_1 \times G_2 \Rightarrow \operatorname{edim}_k G \leq \operatorname{edim}_k G_1 + \operatorname{edim}_k G_2$
- k'/k field extension $\Rightarrow \operatorname{edim}_{k'} G \leq \operatorname{edim}_k G$
- $\operatorname{edim}_k G \leq \operatorname{edim}_k G/N + [G : N] \operatorname{edim}_k N$
- $\operatorname{edim}_k G \wr S_n \leq n \operatorname{edim}_k G$ (where $G \wr S_n = G^n \rtimes S_n$)

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- k'/k field extension $\Rightarrow \mathrm{edim}_{k'} G \leq \mathrm{edim}_k G$
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  $\varphi_G \otimes k' \colon \mathbb{A}(V_G \otimes k') \dashrightarrow \mathbb{A}(W_G \otimes k')$
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- $\operatorname{edim}_k G \leq \operatorname{edim}_k G/N + [G:N] \operatorname{edim}_k N$
  $\varphi_{G/N} \times (\varphi_N)_* \colon \mathbb{A}(V_{G/N} \oplus \operatorname{ind}_N^G V_N) \dashrightarrow \mathbb{A}(W_{G/N} \oplus \operatorname{ind}_N^G W_N)$
  $\operatorname{ind}_H^G V = \{\alpha \colon G \to V \mid \alpha(hx) = h\alpha(x)\}, \quad (\varphi_N)_*(\alpha) = \varphi_N \circ \alpha.$
  Sometimes $\operatorname{edim}_k G/N \gg \operatorname{edim}_k G$.
- $\operatorname{edim}_k G \wr S_n \leq n \operatorname{edim}_k G$ (where $G \wr S_n = G^n \rtimes S_n$)

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- $H \subseteq G$ subgroup $\Rightarrow \operatorname{edim}_k H \leq \operatorname{edim}_k G$
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- $\operatorname{edim}_k G \wr S_n \leq n \operatorname{edim}_k G$ (where $G \wr S_n = G^n \rtimes S_n$)
  $\varphi_G \times \cdots \times \varphi_G \colon \mathbb{A}(V_G^{\oplus n}) \dashrightarrow \mathbb{A}(W_G^{\oplus n})$

Some known results

- BR If A is abelian and $\zeta_{\exp A} \in k$ then $\operatorname{edim}_k A = \operatorname{rk} A$.
- BR If $n \geq 5$ and char $k \neq 2$ then $\lfloor \frac{n}{2} \rfloor \leq \operatorname{edim}_k S_n \leq n - 3$.
- BR If $\operatorname{edim}_k G = 1$ and char $k \nmid |G|$, then G is either cyclic or dihedral of odd order.
- Le If char $k = p > 0$ then $\operatorname{edim}_k \mathbb{Z}/p^n\mathbb{Z} \leq n$.
- Le If $k \supseteq \mathbb{F}_{p^n}$ then $\operatorname{edim}_k(\mathbb{Z}/p\mathbb{Z})^n = 1$.
- KM If G is a p-group and $\zeta_p \in k$ then $\operatorname{edim}_k G = \operatorname{rdim}_k G$.

BR: Buhler and Reichstein, **Le:** Ledet, **KM:** Karpenko and Merkurjev.

Where we are:

Goal:

Find covariants $\varphi\colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(W)$, $\dim \varphi = \operatorname{edim}_k G$ with useful special properties.

- Regular covariants: $\varphi = (f_1, f_2, \dots, f_n)$ with all $f_i \in k[V]$ polynomials.
- Homogeneous covariants: All f_i are homogeneous of the same degree $m \in \mathbb{Z}$ (possibly some $f_i = 0$)
  Equivalently: $\varphi(\lambda v) = \lambda^m \varphi(v)$ for an indeterminate λ.

Why is homogeneity useful?

- One reason: Let $p \neq \operatorname{char} k$ be a prime with $p \nmid m, p \nmid |Z(G)|$. Then φ can be turned into a covariant of $G \times \mathbb{Z}/p\mathbb{Z}$.
- Another reason: If $m \neq 0$ then $X := \overline{\varphi(V)}$ is a cone (i.e. stable under the action $\mathbb{G}_m \times \mathbb{A}(W) \to \mathbb{A}(W), (\lambda, x) \mapsto \lambda x$).

Question:

Does every group G have a faithful homogeneous covariant $\varphi\colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(W)$ of dimension $\mathrm{edim}_k G$?

Answer:

No!

But:

If W is irreducible it works! Replace φ by its highest degree component $\varphi_{\max}\colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(W)$.

- $\varphi_{\max}$ is again a covariant.
- $\varphi_{\max}$ is faithful.
- $\dim \varphi_{\max} \leq \dim \varphi$.

Problem:

Not every group admits a faithful simple representation!

Multihomogeneous covariants

Assume that char $k = 0$ or G does not contain any nontrivial normal p-subgroup if char $k = p > 0$. Then G admits a faithful semi-simple representation!

- $V = \bigoplus_{i=1}^{m} V_i, \quad V_i$ irreducible.
- $W = \bigoplus_{j=1}^{n} W_j, \quad W_j$ irreducible.
- $\varphi = (\varphi_1, \ldots, \varphi_n) \colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(W)$ (where $\varphi_j \colon \mathbb{A}(V) \to \mathbb{A}(W_j)$) is called multihomogeneous if for some $m_{ij} \in \mathbb{Z}$

$$\varphi_j(v_1, \ldots, \lambda v_i, \ldots v_n) = \lambda^{m_{ij}} \varphi_j(v_1, \ldots, v_n).$$

- $M_\varphi := (m_{ij}) \in \mathrm{Mat}_{m \times n}(\mathbb{Z})$ gives useful information!
- Faithfulness is tested easily: $\varphi = (\varphi_1, \ldots, \varphi_n)$ is faithful if all $\varphi_i \neq 0$!

A simple application

- $G = (\mathbb{Z}/p\mathbb{Z})^n$.
- k a field with a primitive p-th root of unity ζ.
- V canonical n-dim. faithful G-module.
- $\varphi\colon \mathbb{A}(V) \to \mathbb{A}(V)$ multihomogeneous $\Rightarrow$

$$\begin{aligned}(\zeta\varphi_1, \varphi_2, \dots, \varphi_n)(v) &= \varphi(\zeta v_1, \dots, v_n) \\ &= (\zeta^{m_{11}}\varphi_1, \zeta^{m_{12}}\varphi_2, \dots, \zeta^{m_{1n}}\varphi_n)(v)\end{aligned}$$

- It follows that $M_\varphi \equiv I_n \pmod{p}$.
- In particular: $\det M_\varphi \equiv 1 \pmod{p}$ and M_φ has rank n.
- One finds a torus T of dimension $\operatorname{rk} M_\varphi$ acting freely on a dense open subset of $\overline{\varphi(V)}$. Therefore $\dim \varphi = n$.

Multihomogenization

- G a group admitting semi-simple faithful G-modules $V = \bigoplus_{i=1}^{m} V_i$, $W = \bigoplus_{j=1}^{n} W_j$
- $\varphi\colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(W)$ an arbitrary covariant.
- There is a covariant $H_\lambda(\varphi)\colon \mathbb{A}(V) \to \mathbb{A}(W)$ depending only on φ and the choice of a suitable grading λ on $k[W]$ s.t.:
  - $H_\lambda(\varphi)$ is multihomogeneous.
  - $\dim H_\lambda(\varphi) \leq \dim \varphi$.
  - $\pi_j \circ H_\lambda(\varphi) \neq 0 \Leftrightarrow \pi_j \circ \varphi \neq 0$, where $\pi_j\colon \mathbb{A}(W) \to \mathbb{A}(W_j)$.

Thus: $\operatorname{edim}_k G = \min\{\dim \varphi \mid \varphi$ multihomogeneous covariant$\}$.

Theorem about the rank of M_φ:

Let $Z(G,k)$ denote the subgroup $\{g \in Z(G) \mid \zeta_{\operatorname{ord} g} \in k\} \subseteq Z(G)$.

- $\operatorname{rk} M_\varphi \geq \operatorname{rk} Z(G,k)$.
- $\operatorname{edim}_k G \leq \dim \varphi - (\operatorname{rk} M_\varphi - \operatorname{rk} Z(G,k))$.

Where we are:

Direct products

- $G = G_1 \times G_2$.
- $\varphi_i \colon \mathbb{A}(V_i) \dashrightarrow \mathbb{A}(W_i)$ minimal faithful covariant of G_i, $i = 1, 2$
- $\varphi := \varphi_1 \times \varphi_2 \colon \mathbb{A}(V_1 \oplus V_2) \dashrightarrow \mathbb{A}(W_1 \oplus W_2)$ is again a faithful covariant, $\mathrm{edim}_k\, G \leq \mathrm{edim}_k\, G_1 + \mathrm{edim}_k\, G_2$.
- Now if φ_1, φ_2 are multihomogeneous, then so is φ and

$$\mathrm{rk}\, M_\varphi = \mathrm{rk}\, M_{\varphi_1} + \mathrm{rk}\, M_{\varphi_2} \overset{Th}{=} \mathrm{rk}\, Z(G_1, k) + \mathrm{rk}\, Z(G_2, k),$$
$$\dim \varphi = \dim \varphi_1 + \dim \varphi_2 = \mathrm{edim}_k\, G_1 + \mathrm{edim}_k\, G_2.$$

$$\begin{aligned} \mathrm{edim}_k\, G_1 \times G_2 &\overset{Th}{\leq} \dim \varphi - (\mathrm{rk}\, M_{\varphi_1 \times \varphi_2} - \mathrm{rk}\, Z(G, k)) \\ &= \mathrm{edim}_k\, G_1 + \mathrm{edim}_k\, G_2 - \\ &- (\mathrm{rk}\, Z(G_1, k) + \mathrm{rk}\, Z(G_2, k) - \mathrm{rk}\, Z(G, k)). \end{aligned}$$

Subgroups

- Let $H \subseteq G$ be a subgroup. Assume that G admits a faithful semi-simple G-module $V = \bigoplus V_i$ s.t. $V\downarrow_H$ is semi-simple, too.
- Let $\varphi\colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(V)$ be a multihomogeneous minimal faithful covariant. So $\dim\varphi = \operatorname{edim}_k G$, $\operatorname{rk} M_\varphi \overset{Th}{=} \operatorname{rk} Z(G,k)$.
- Regard φ as a H-covariant. Refine decomposition of $V\downarrow_H = \bigoplus (V_i)\downarrow_H$ into irreducibles.
- "Multihomogenization" yields a faithful multihomogeneous H-covariant $\tilde{\varphi}\colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(V)$ with $\dim\tilde{\varphi} \leq \dim\varphi$ and $\operatorname{rk} M_{\tilde{\varphi}} \geq \operatorname{rk} M_\varphi$.

$$\begin{aligned}
\operatorname{edim}_k H &\overset{Th}{\leq} \dim\tilde{\varphi} - (\operatorname{rk} M_{\tilde{\varphi}} - \operatorname{rk} Z(H,k)) \\
&\leq \dim\varphi - (\operatorname{rk} M_\varphi - \operatorname{rk} Z(H,k)) \\
&= \operatorname{edim}_k G - (\operatorname{rk} Z(G,k) - \operatorname{rk} Z(H,k)).
\end{aligned}$$

Central extensions

- Let $H \subseteq Z(G)$ such that $H \cap [G, G] = \{e\}$ and assume that k contains a suitable root of unity.
- Then G has a faithful semi-simple G-module of the form

$$V = W \oplus k_{\chi_1} \oplus \cdots \oplus k_{\chi_r},$$

where $\chi_i \in \mathrm{Hom}(G, k)$, $r = \mathrm{rk}\, H$ and H acts trivially on W.

- Let $\varphi\colon \mathbb{A}(W) \dashrightarrow \mathbb{A}(W)$ be a multihomogeneous minimal faithful covariant for G/H.
- $\Phi\colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(V), \quad (w, t_1, \dots, t_r) \mapsto (\varphi(w), t_1, \dots, t_r).$
- Φ is a faithful multihomogeneous G-covariant and $\mathrm{rk}\, M_\Phi = \mathrm{rk}\, M_\varphi + r \overset{Th}{=} \mathrm{rk}\, Z(G/H, k) + r.$

$$\begin{aligned} \mathrm{edim}_k\, G &\overset{Th}{\leq} \dim \Phi - (\mathrm{rk}\, M_\Phi - \mathrm{rk}\, Z(G, k)) \\ &= \mathrm{edim}_k\, G/H - \mathrm{rk}\, Z(G/H, k) + \mathrm{rk}\, Z(G, k). \end{aligned}$$

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- Let $\varphi \colon \mathbb{A}(W) \dashrightarrow \mathbb{A}(W)$ be a multihomogeneous minimal faithful covariant for G/H.
- $\Phi \colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(V), \quad (w, t_1, \ldots, t_r) \mapsto (\varphi(w), t_1, \ldots, t_r).$

$$\begin{aligned} \mathrm{edim}_k\, G &\overset{Th}{\leq} \dim \Phi - (\mathrm{rk}\, M_\Phi - \mathrm{rk}\, Z(G, k)) \\ &= \mathrm{edim}_k\, G/H - \mathrm{rk}\, Z(G/H, k) + \mathrm{rk}\, Z(G, k). \end{aligned}$$

- The reversed inequality also holds!
- This generalizes results of Buhler/Reichstein (1997), Kang (2006), Kraft/Schwarz (2007).

Where we are:

Generalization of M. Florence's twisting technique

- Let $V = \bigoplus_{i=1}^{n} V_i$ be a semi-simple faithful G-module.
- Let $\mathbb{P}(V) := \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_n)$.
- $\pi\colon \mathbb{A}(V) \dashrightarrow \mathbb{P}(V)$ the rational quotient map.
- Let $\varphi\colon \mathbb{A}(V) \dashrightarrow \mathbb{A}(V)$ be a (minimal) faithful multihomogeneous covariant.
- Get a rational map $\psi\colon \mathbb{P}(V) \dashrightarrow \mathbb{P}(V)$ fitting into a commutative square:

$$\begin{array}{ccc} \mathbb{A}(V) & \overset{\varphi}{\dashrightarrow} & \mathbb{A}(V) \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi} \\ \mathbb{P}(V) & \overset{\psi}{\dashrightarrow} & \mathbb{P}(V) \end{array}$$

- ψ is G/Z-equivariant where $Z := Z(G,k) = \{g \in Z(G) | \zeta_{\mathrm{ord}\, g} \in k\}$.

Generalization of M. Florence's twisting technique

- $$\begin{array}{ccc} \mathbb{A}(V) & \overset{\varphi}{\dashrightarrow} & \mathbb{A}(V) \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{P}(V) & \overset{\psi}{\dashrightarrow} & \mathbb{P}(V) \end{array}$$
- ψ is G/Z-equivariant where $Z := Z(G, k)$
- **Idea:** Twist $\psi \colon \mathbb{P}(V) \dashrightarrow \mathbb{P}(V)$ with a G/Z-torsor $E \to \operatorname{spec} K$, obtain a rational map

  $${}^{E}\psi\colon \operatorname{SB}(A_1) \times \cdots \times \operatorname{SB}(A_n) \dashrightarrow \operatorname{SB}(A_1) \times \cdots \times \operatorname{SB}(A_n),$$

  where $A_i = {}^{E}(\operatorname{End} V_i \otimes_k K)$ is a central simple K-algebra.
- **Fact:** Rational maps $\prod \operatorname{SB}(A_i) \dashrightarrow \prod \operatorname{SB}(A_i)$ are dominant in specific cases (by results of Karpenko and Merkurjev).
- In these cases:

  $$\operatorname{edim}_k G - \operatorname{rk} Z(G,k) = \dim \varphi - \operatorname{rk} Z(G,k) = \dim \psi$$
  $$\geq \dim {}^{E}\psi = \sum_i \dim \operatorname{SB}(A_i) = \dim \mathbb{P}(V) = \dim V - n.$$

Central simple algebras and Severi-Brauer varieties

Central simple algebras

- A K-algebra A is called central simple if $Z(A) = K$ and A has no nontrivial two-sided ideals.
- Central simple algebras are precisely the "twisted forms" of matrix algebras (i.e. $A \otimes_K \overline{K} \simeq M_n(\overline{K})$).

Severi-Brauer varieties

Let A be a central simple K-algebra of dimension n^2.

- The Severi-Brauer variety $\mathrm{SB}(A)$ is the projective variety defined by $\mathrm{SB}(A)(L) = \{n\text{-dim. right ideals of } A \otimes_K L\}$ for L/K field extension.
- $\mathrm{SB}(A)(L) \neq \emptyset \Leftrightarrow A \otimes_K L \simeq M_n(L)$
- Severi-Brauer varieties are precisely the "twisted forms" of projective space $\mathbb{P}^{n-1}$.

Twisting

- Let A a central simple K-algebra with an action of a group H.
- Let E be a H-torsor over K.
- $E = \text{spec}(L)$ where L/K is a Galois H-algebra (i.e. an étale algebra of dimension $|H|$ with $L^H = K$).
- ${}^E A := (L \otimes A)^H$ is a central simple K-algebra, the "twist" of A by E, where H acts on $L \otimes_K A$ by $h(l \otimes a) = hl \otimes ha$.
- Twist of quasi-projective H-varieties over K: ${}^E X = (E \times X)/H$, where H acts on $E \times X$ by $h(e, x) = (eh^{-1}, hx)$.
- ${}^E \text{SB}(A) \simeq \text{SB}({}^E A)$.
- A H-equivariant rational map $\varphi\colon X \dashrightarrow Y$ induces a rational map ${}^E\varphi\colon {}^E X \dashrightarrow {}^E Y$.
- Functoriality: ${}^E(\psi \circ \varphi) = {}^E\psi \circ {}^E\varphi$.

Result

Let $C \subseteq Z(G)$ be a subgroup. Let for $\chi \in C^* := \mathrm{Hom}(C, k)$:

$$\mathrm{Rep}^{(\chi)}(G) := \{V \text{ irreducible } G\text{-module } | \forall g \in C, v \in V : gv = \chi(g)v\},$$

$$D^{(\chi)}(G) := \{\dim V \mid V \in \mathrm{Rep}^{(\chi)}(G)\}.$$

Theorem (Generalization of Karpenko and Merkurjev's Theorem)

Let G be a group whose socle $C := \mathrm{soc}\, G$ is a central subgroup of p-power order and assume $\zeta_p \in k$. If for all $\chi \in C^*$

$$\gcd D^{(\chi)}(G) = \min D^{(\chi)}(G),$$

then $\mathrm{edim}_k\, G = \min\{\dim V | V \text{ faithful } G\text{-module}\}$.

The proof relies heavily on results of **Karpenko** and **Merkurjev** on rational maps of Severi-Brauer varieties and their calculation of the index of the twist of $\mathrm{End}(V_i)$ by a generic G/Z-torsor.

Result

$$D^{(\chi)}(G) := \{\dim V \mid V \in \mathrm{Rep}^{(\chi)}(G)\}.$$

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Examples

- G a p-group: $\mathrm{soc}\, G = \{g \in Z(G) \mid g^p = e\}$, $D^{(\chi)}(G) \subset \{p^r | r \in \mathbb{N}_0\}$.
- G = nonsplit central extension of A_8 by C_2: $\mathrm{soc}\, G = C_2$, $D^{(\chi)}(G) = \{8, 24, 48\}$ (char $k = 3$) where $\chi \neq 1$.